Higher Dimensional Algebra v.5

Biographies

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Abstract algebra

Abstract algebra is the subject area of mathematics that studies algebraic structures, such as groups, rings, fields, modules, vector spaces, and algebras. The phrase **abstract algebra** was coined at the turn of the 20th century to distinguish this area from what was normally referred to as **algebra**, the study of the rules for manipulating formulae and algebraic expressions involving unknowns and real or complex numbers, often now called *elementary algebra*. The distinction is rarely made in more recent writings.

Contemporary mathematics and mathematical physics make intensive use of abstract algebra; for example, theoretical physics draws on Lie algebras. Subject areas such as algebraic number theory, algebraic topology, and algebraic geometry apply algebraic methods to other areas of mathematics. Representation theory, roughly speaking, takes the 'abstract' out of 'abstract algebra', studying the concrete side of a given structure; see model theory.

Two mathematical subject areas that study the properties of algebraic structures viewed as a whole are universal algebra and category theory. Algebraic structures, together with the associated homomorphisms, form categories. Category theory is a powerful formalism for studying and comparing different algebraic structures.

History and examples

As in other parts of mathematics, concrete problems and examples have played important roles in the development of algebra. Through the end of the nineteenth century many, perhaps most of these problems were in some way related to the theory of algebraic equations. Among major themes we can mention:

- solving of systems of linear equations, which led to matrices, determinants and linear algebra.
- attempts to find formulae for solutions of general polynomial equations of higher degree that resulted in discovery of groups as abstract manifestations of symmetry;
- and arithmetical investigations of quadratic and higher degree forms and diophantine equations, notably, in proving Fermat's last theorem, that directly produced the notions of a ring and ideal.

Numerous textbooks in abstract algebra start with axiomatic definitions of various algebraic structures and then proceed to establish their properties, creating a false impression that somehow in algebra axioms had come first and then served as a motivation and as a basis of further study. The true order of historical development was almost exactly the opposite. Most theories that we now recognize as parts of algebra started as collections of disparate facts from various branches of mathematics, acquired a common theme that served as a core around which various results were grouped, and finally became unified on a basis of a common set of concepts. An archetypical example of this progressive synthesis can be seen in the theory of groups.

Early group theory

There were several threads in the early development of group theory, in modern language loosely corresponding to *number theory*, *theory of equations*, and *geometry*, of which we concentrate on the first two.

Leonhard Euler considered algebraic operations on numbers modulo an integer, modular arithmetic, proving his generalization of Fermat's little theorem. These investigations were taken much further by Carl Friedrich Gauss, who considered the structure of multiplicative groups of residues mod n and established many properties of cyclic and more general abelian groups that arise in this way. In his investigations of composition of binary quadratic forms, Gauss explicitly stated the associative law for the composition of forms, but like Euler before him, he seems to have been more interested in concrete results than in general theory. In 1870, Leopold Kronecker gave a definition of an abelian group in the context of ideal class groups of a number field, a far-reaching generalization of Gauss's work. It appears that he did not tie it with previous work on groups, in particular, permutation groups. In 1882 considering the same question, Heinrich M. Weber realized the connection and gave a similar definition that involved the cancellation property but omitted the existence of the inverse element, which was sufficient in his context (finite groups).

Permutations were studied by Joseph Lagrange in his 1770 paper *Réflexions sur la résolution algébrique des équations* devoted to solutions of algebraic equations, in which he introduced Lagrange resolvents. Lagrange's goal was to understand why equations of third and fourth degree admit formulae for solutions, and he identified as key objects permutations of the roots. An important novel step taken by Lagrange in this paper was the abstract view of the roots, i.e. as symbols and not as numbers. However, he did not consider composition of permutations. Serendipitously, the first edition of Edward Waring's *Meditationes Algebraicae* appeared in the same year, with an expanded version published in 1782. Waring proved the main theorem on symmetric functions, and specially considered the relation between the roots of a quartic equation and its resolvent cubic. *Mémoire sur la résolution des équations* of Alexandre Vandermonde (1771) developed the theory of symmetric functions from a slightly different angle, but like Lagrange, with the goal of understanding solvability of algebraic equations.

> *Kronecker claimed in 1888 that the study of modern algebra began with this first paper of Vandermonde. Cauchy states quite clearly that Vandermonde had priority over Lagrange for this remarkable idea which eventually led to the study of group theory.* [1]

Paolo Ruffini was the first person to develop the theory of permutation groups, and like his predecessors, also in the context of solving algebraic equations. His goal was to establish impossibility of algebraic solution to a general algebraic equation of degree greater than four. En route to this goal he introduced the notion of the order of an element of a group, conjugacy, the cycle decomposition of elements of permutation groups and the notions of primitive and imprimitive and proved some important theorems relating these concepts, such as

> *if G is a subgroup of S_5 whose order is divisible by 5 then G contains an element of order 5.*

Note, however, that he got by without formalizing the concept of a group, or even of a permutation group. The next step was taken by Évariste Galois in 1832, although his work remained unpublished until 1846, when he considered for the first time what we now call the *closure property* of a group of permutations, which he expressed as

> ... if in such a group one has the substitutions S and T then one has the substitution ST.

The theory of permutation groups received further far-reaching development in the hands of Augustin Cauchy and Camille Jordan, both through introduction of new concepts and, primarily, a great wealth of results about special classes of permutation groups and even some general theorems. Among other things, Jordan defined a notion of isomorphism, still in the context of permutation groups and, incidentally, it was he who put the term *group* in wide use.

The abstract notion of a group appeared for the first time in Arthur Cayley's papers in 1854. Cayley realized that a group need not be a permutation group (or even *finite*), and may instead consist of matrices, whose algebraic properties, such as multiplication and inverses, he systematically investigated in succeeding years. Much later Cayley would revisit the question whether abstract groups were more general than permutation groups, and establish that, in fact, any group is isomorphic to a group of permutations.

Modern algebra

The end of 19th and the beginning of the 20th century saw a tremendous shift in methodology of mathematics. No longer satisfied with establishing properties of concrete objects, mathematicians started to turn their attention to general theory. For example, results about various groups of permutations came to be seen as instances of general theorems that concern a general notion of an *abstract group*. Questions of structure and classification of various mathematical objects came to forefront. These processes were occurring throughout all of mathematics, but became especially pronounced in algebra. Formal definition through primitive operations and axioms were proposed for many basic algebraic structures, such as groups, rings, and fields. The algebraic investigations of general fields by Ernst Steinitz and of commutative and then general rings by David Hilbert, Emil Artin and Emmy Noether, building up on the work of Ernst Kummer, Leopold Kronecker and Richard Dedekind, who had considered ideals in commutative rings, and of Georg Frobenius and Issai Schur, concerning representation theory of groups, came to define abstract algebra. These developments of the last quarter of the 19th century and the first quarter of 20th century were systematically exposed in Bartel van der Waerden's *Moderne algebra*, the two-volume monograph published in 1930–1931 that forever changed for the mathematical world the meaning of the word *algebra* from *the theory of equations* to the *theory of algebraic structures*.

An example

Abstract algebra facilitates the study of properties and patterns that seemingly disparate mathematical concepts have in common. For example, consider the distinct operations of function composition, $f(g(x))$, and of matrix multiplication, AB. These two operations have, in fact, the same structure. To see this, think about multiplying two square matrices, AB, by a one column vector, x. This defines a function equivalent to composing Ay with Bx: $Ay = A(Bx) = (AB)x$. Functions under composition and matrices under multiplication are examples of monoids. A set S and a binary operation over S, denoted by concatenation, form a monoid if the operation associates, $(ab)c = a(bc)$, and if there exists an $e \in S$, such that $ae = ea = a$.

See also

- Universal algebra
- Coding theory
- Important publications in abstract algebra

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External links

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- Joseph Mileti: *Mathematics Museum: Abstract Algebra* [8], A good introduction to the subject in real-life terms.

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[5] http://www.math.miami.edu/~ec/book/
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[8] http://www.math.uchicago.edu/~mileti/museum/algebra.html

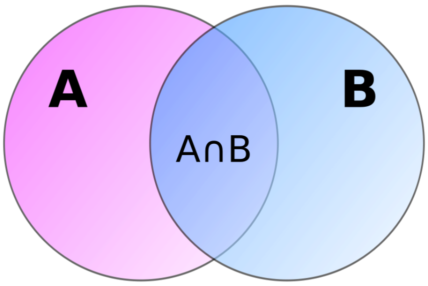
Set (mathematics)

A **set** is a collection of distinct objects, considered as an object in its own right. Sets are one of the most fundamental concepts in mathematics. Although it was invented at the end of the 19th century, set theory is now a ubiquitous part of mathematics, and can be used as a foundation from which nearly all of mathematics can be derived. In mathematics education, elementary topics such as Venn diagrams are taught at a young age, while more advanced concepts are taught as part of a university degree.

Definition

Georg Cantor, the founder of set theory, gave the following definition of a set at the beginning of his *Beiträge zur Begründung der transfiniten Mengenlehre*:[1]

> By a "set" we mean any collection *M* into a whole of definite, distinct objects *m* (which are called the "elements" of *M*) of our perception [Anschauung] or of our thought.



The *intersection* of two sets is made up of the objects contained in both sets, shown in a Venn diagram.

The elements or members of a set can be anything: numbers, people, letters of the alphabet, other sets, and so on. Sets are conventionally denoted with capital letters. Sets *A* and *B* are equal if and only if they have precisely the same elements.

As discussed below, in formal mathematics the definition given above turned out to be inadequate; instead, the notion of a "set" is taken as an undefined primitive in axiomatic set theory, and its properties are defined by the Zermelo–Fraenkel axioms. The most basic properties are that a set "has" elements, and that two sets are equal (one and the same) if they have the same elements.

Describing sets

There are two ways of describing, or specifying the members of, a set. One way is by intensional definition, using a rule or semantic description:

A is the set whose members are the first four positive integers.

B is the set of colors of the French flag.

The second way is by extension – that is, listing each member of the set. An extensional definition is denoted by enclosing the list of members in brackets:

$C = \{4, 2, 1, 3\}$

$D = \{\text{blue, white, red}\}$

Unlike a multiset, every element of a set must be unique; no two members may be identical. All set operations preserve the property that each element of the set is unique. The order in which the elements of a set are listed is irrelevant, unlike a sequence or tuple. For example,

$\{6, 11\} = \{11, 6\} = \{11, 11, 6, 11\}$,

because the extensional specification means merely that each of the elements listed is a member of the set.

For sets with many elements, the enumeration of members can be abbreviated. For instance, the set of the first thousand positive integers may be specified extensionally as:

$\{1, 2, 3, ..., 1000\}$,

where the ellipsis ("...") indicates that the list continues in the obvious way. Ellipses may also be used where sets have infinitely many members. Thus the set of positive even numbers can be written as $\{2, 4, 6, 8, ... \}$.

The notation with braces may also be used in an intensional specification of a set. In this usage, the braces have the meaning "the set of all ...". So, E = {playing card suits} is the set whose four members are ♠, ♦, ♥, and ♣. A more general form of this is set-builder notation, through which, for instance, the set F of the twenty smallest integers that are four less than perfect squares can be denoted:

$F = \{n^2 - 4 : n \text{ is an integer; and } 0 \le n \le 19\}$

In this notation, the colon (":") means "such that", and the description can be interpreted as "F is the set of all numbers of the form $n^2 - 4$, such that n is a whole number in the range from 0 to 19 inclusive." Sometimes the vertical bar ("|") is used instead of the colon.

One often has the choice of specifying a set intensionally or extensionally. In the examples above, for instance, $A = C$ and $B = D$.

Membership

The key relation between sets is *membership* – when one set is an element of another. If A is a member of B, this is denoted $A \in B$, while if C is not a member of B then $C \notin A$. For example, with respect to the sets $A = \{1,2,3,4\}$, B = {blue, white, red}, and $F = \{n^2 - 4 : n \text{ is an integer; and } 0 \le n \le 19\}$ defined above,

$4 \in A$ and $285 \in F$; but

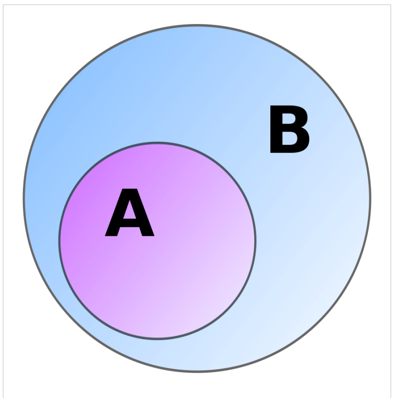
$9 \notin F$ and green $\notin B$.

Subsets

If every member of set A is also a member of set B, then A is said to be a *subset* of B, written $A \subseteq B$ (also pronounced *A is contained in B*). Equivalently, we can write $B \supseteq A$, read as *B is a superset of A*, *B includes A*, or *B contains A*. The relationship between sets established by $\subseteq$ is called *inclusion* or *containment*.

If A is a subset of, but not equal to, B, then A is called a *proper subset* of B, written $A \subsetneq B$ (*A is a proper subset of B*) or $B \supsetneq A$ (*B is proper superset of A*).

Note that the expressions $A \subset B$ and $A \supset B$ are used differently by different authors; some authors use them to mean the same as $A \subseteq B$ (respectively $A \supseteq B$), whereas other use them to mean the same as $A \subsetneq B$ (respectively $A \supsetneq B$).



A is a **subset** of *B*

Example:

- The set of all men is a proper subset of the set of all people.
- $\{1, 3\} \subsetneq \{1, 2, 3, 4\}$.
- $\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$.

The empty set is a subset of every set and every set is a subset of itself:

- $\varnothing \subseteq A$.
- $A \subseteq A$.

An obvious but useful identity, which can often be used to show that two seemingly different sets are equal:

- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Power sets

The power set of a set S is the set of all subsets of S. This includes the subsets formed from all the members of S and the empty set. If a finite set S has cardinality n then the power set of S has cardinality 2^n. The power set can be written as $P(S)$.

If S is an infinite (either countable or uncountable) set then the power set of S is always uncountable. Moreover, if S is a set, then there is never a bijection from S onto $P(S)$. In other words, the power set of S is always strictly "bigger" than S.

As an example, the power set $P(\{1, 2, 3\})$ of $\{1, 2, 3\}$ is $\{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \varnothing\}$. The cardinality of the original set is 3, and the cardinality of the power set is $2^3 = 8$. This relationship is one of the reasons for the terminology *power set*. Similarly, its notation is an example of a general convention providing notations for sets based on their cardinalities.

Cardinality

The cardinality | S | of a set S is "the number of members of S." For example, since the French flag has three colors, | B | = 3.

There is a unique set with no members and zero cardinality, which is called the *empty set* (or the *null set*) and is denoted by the symbol ∅ (other notations are used; see empty set). For example, the set of all three-sided squares has zero members and thus is the empty set. Though it may seem trivial, the empty set, like the number zero, is important in mathematics; indeed, the existence of this set is one of the fundamental concepts of axiomatic set theory.

Some sets have infinite cardinality. The set **N** of natural numbers, for instance, is infinite. Some infinite cardinalities are greater than others. For instance, the set of real numbers has greater cardinality than the set of natural numbers. However, it can be shown that the cardinality of (which is to say, the number of points on) a straight line is the same as the cardinality of any segment of that line, of the entire plane, and indeed of any finite-dimensional Euclidean space.

Special sets

There are some sets which hold great mathematical importance and are referred to with such regularity that they have acquired special names and notational conventions to identify them. One of these is the empty set. Many of these sets are represented using blackboard bold or bold typeface. Special sets of numbers include:

- **P**, denoting the set of all primes: **P** = {2, 3, 5, 7, 11, 13, 17, ...}.
- **N**, denoting the set of all natural numbers: **N** = {1, 2, 3, . . .}. (Sometimes **N** = {0, 1, 2, 3, ...}).
- **Z**, denoting the set of all integers (whether positive, negative or zero): **Z** = {... , −2, −1, 0, 1, 2, ...}.
- **Q**, denoting the set of all rational numbers (that is, the set of all proper and improper fractions): **Q** = {a/b : a, b ∈ **Z**, b ≠ 0}. For example, 1/4 = ∈ **Q** and 11/6 ∈ **Q**. All integers are in this set since every integer a can be expressed as the fraction $a/1$.
- **R**, denoting the set of all real numbers. This set includes all rational numbers, together with all irrational numbers (that is, numbers which cannot be rewritten as fractions, such as π, e, and √2.
- **C**, denoting the set of all complex numbers: **C** = {$a + bi$: a, b ∈ **R**}. For example, $1 + 2i$ ∈ **C**.
- **H**, denoting the set of all quaternions: **H** = {$a + bi + cj + dk$: a, b, c, d ∈ **R**}. For example, $1 + i + 2j - k$ ∈ **H**.

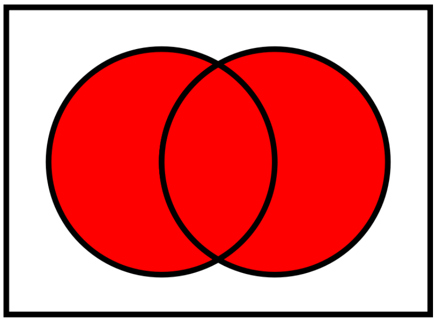
Each of the above sets of numbers has an infinite number of elements, and each can be considered to be a proper subset of the sets listed below it. The primes are used less frequently than the others outside of number theory and related fields.

Basic operations

There are several fundamental operations for constructing new sets from given sets.

Unions

Two sets can be "added" together. The *union* of A and B, denoted by $A \cup B$, is the set of all things which are members of either A or B.



The union of A and B, denoted A ∪ B

Examples:

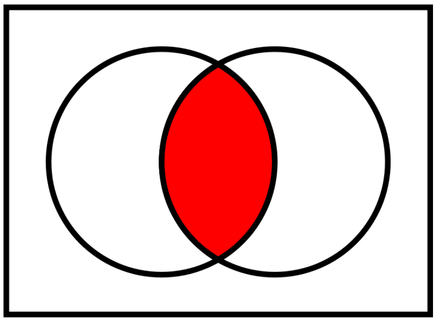
- $\{1, 2\} \cup \{\text{red, white}\} = \{1, 2, \text{red, white}\}$.
- $\{1, 2, \text{green}\} \cup \{\text{red, white, green}\} = \{1, 2, \text{red, white, green}\}$.
- $\{1, 2\} \cup \{1, 2\} = \{1, 2\}$.

Some basic properties of unions:

- $A \cup B = B \cup A$.
- $A \cup (B \cup C) = (A \cup B) \cup C$.
- $A \subseteq (A \cup B)$.
- $A \cup A = A$.
- $A \cup \varnothing = A$.
- $A \subseteq B$ if and only if $A \cup B = B$.

Intersections

A new set can also be constructed by determining which members two sets have "in common". The *intersection* of A and B, denoted by $A \cap B$, is the set of all things which are members of both A and B. If $A \cap B = \varnothing$, then A and B are said to be *disjoint*.



The intersection of A and B, denoted A ∩ B.

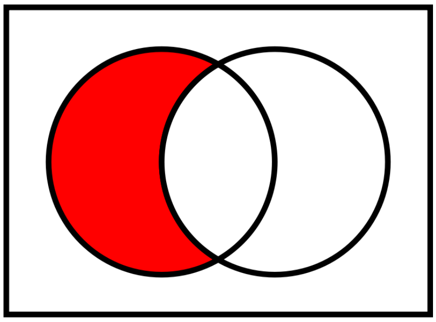
Examples:

- $\{1, 2\} \cap \{\text{red, white}\} = \varnothing$.
- $\{1, 2, \text{green}\} \cap \{\text{red, white, green}\} = \{\text{green}\}$.
- $\{1, 2\} \cap \{1, 2\} = \{1, 2\}$.

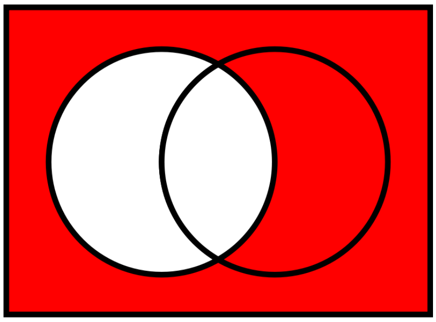
Some basic properties of intersections:

- $A \cap B = B \cap A$.
- $A \cap (B \cap C) = (A \cap B) \cap C$.
- $A \cap B \subseteq A$.
- $A \cap A = A$.
- $A \cap \varnothing = \varnothing$.
- $A \subseteq B$ if and only if $A \cap B = A$.

Complements


The relative complementof B in A

Two sets can also be "subtracted". The *relative complement* of B in A (also called the *set-theoretic difference* of A and B), denoted by $A \setminus B$, (or $A - B$) is the set of all elements which are members of A but not members of B. Note that it is valid to "subtract" members of a set that are not in the set, such as removing the element *green* from the set {1, 2, 3}; doing so has no effect.


The complement of A in U

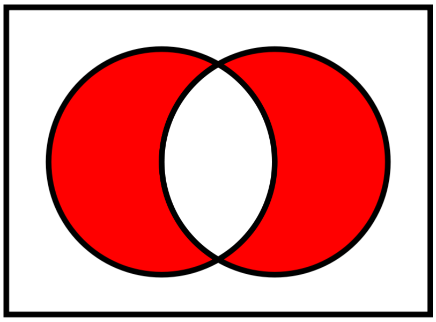
In certain settings all sets under discussion are considered to be subsets of a given universal set U. In such cases, $U \setminus A$ is called the *absolute complement* or simply *complement* of A, and is denoted by A'.

Examples:

- {1, 2} \ {red, white} = {1, 2}.
- {1, 2, green} \ {red, white, green} = {1, 2}.
- {1, 2} \ {1, 2} = ∅.
- {1, 2, 3, 4} \ {1, 3} = {2, 4}.
- If U is the set of integers, E is the set of even integers, and O is the set of odd integers, then $E' = O$.

Some basic properties of complements:

- $A \setminus B \neq B \setminus A$.
- $A \cup A' = U$.
- $A \cap A' = \varnothing$.
- $(A')' = A$.
- $A \setminus A = \varnothing$.
- $U' = \varnothing$ and $\varnothing' = U$.
- $A \setminus B = A \cap B'$.


The symmetric difference of A and B

An extension of the complement is the symmetric difference, defined for sets A, B as

$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A).$$

For example, the symmetric difference of {7,8,9,10} and {9,10,11,12} is the set {7,8,11,12}.

Cartesian product

A new set can be constructed by associating every element of one set with every element of another set. The *Cartesian product* of two sets A and B, denoted by $A \times B$ is the set of all ordered pairs (a, b) such that a is a member of A and b is a member of B.

Examples:

- {1, 2} × {red, white} = {(1, red), (1, white), (2, red), (2, white)}.
- {1, 2, green} × {red, white, green} = {(1, red), (1, white), (1, green), (2, red), (2, white), (2, green), (green, red), (green, white), (green, green)}.
- {1, 2} × {1, 2} = {(1, 1), (1, 2), (2, 1), (2, 2)}.

Some basic properties of cartesian products:

- $A \times \varnothing = \varnothing$.
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
- $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

Let A and B be finite sets. Then

- $| A \times B | = | B \times A | = | A | \times | B |$.

Applications

Set theory is seen as the foundation from which virtually all of mathematics can be derived. For example, structures in abstract algebra, such as groups, fields and rings, are sets closed under one or more operations.

One of the main applications of naive set theory is constructing relations. A relation from a domain A to a codomain B is a subset of the cartesian product $A \times B$. Given this concept, we are quick to see that the set F of all ordered pairs (x, x^2), where x is real, is quite familiar. It has a domain set **R** and a codomain set that is also **R**, because the set of all squares is subset of the set of all reals. If placed in functional notation, this relation becomes $f(x) = x^2$. The reason these two are equivalent is for any given value, y that the function is defined for, its corresponding ordered pair, (y, y^2) is a member of the set F.

Axiomatic set theory

Although initially naive set theory, which defines a set merely as *any well-defined* collection, was well accepted, it soon ran into several obstacles. It was found that this definition spawned several paradoxes, most notably:

- Russell's paradox—It shows that the "set of all sets which *do not contain themselves*," i.e. the "set" $\{ x : x$ is a set and $x \notin x \}$ does not exist.
- Cantor's paradox—It shows that "the set of all sets" cannot exist.

The reason is that the phrase *well-defined* is not very well defined. It was important to free set theory of these paradoxes because nearly all of mathematics was being redefined in terms of set theory. In an attempt to avoid these paradoxes, set theory was axiomatized based on first-order logic, and thus **axiomatic set theory** was born.

For most purposes however, naive set theory is still useful.

See also

- Alternative set theory
- Axiomatic set theory
- Class (set theory)
- Dense set
- Family (mathematics)
- Fuzzy set
- Internal set
- Mathematical structure
- Multiset
- Naive set theory
- Rough set
- Russell's paradox
- Scientific classification
- Category of sets
- Taxonomy
- Tuple

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External links

- C2 Wiki - Examples of set operations using English operators. [2]

References

[1] Quoted in Dauben, p. 170.

[2] http://www.c2.com/cgi/wiki?SetTheory

Binary operation

In mathematics, a **binary operation** is a calculation involving two operands, in other words, an operation whose arity is two. Examples include the familiar arithmetic operations of addition, subtraction, multiplication and division.

More precisely, a binary operation on a set S is a binary relation that maps elements of the Cartesian product $S \times S$ to S:

$$f \colon S \times S \rightarrow S.$$

If f is not a function, but is instead a partial function, it is called a **partial operation**. For instance, division of real numbers is a partial function, because one can't divide by zero: $a/0$ is not defined for any real a. Note however that both in algebra and model theory the binary operations considered are defined on the whole of $S \times S$.

Sometimes, especially in computer science, the term is used for any binary function. That f takes values in the same set S that provides its arguments is the property of closure.

Binary operations are the keystone of algebraic structures studied in abstract algebra: they form part of groups, monoids, semigroups, rings, and more. Most generally, a *magma* is a set together with any binary operation defined on it.

Many binary operations of interest in both algebra and formal logic are commutative or associative. Many also have identity elements and inverse elements. Typical examples of binary operations are the addition (+) and multiplication (×) of numbers and matrices as well as composition of functions on a single set.

An example of an operation that is not commutative is subtraction (−). Examples of partial operations that are not commutative include division (/), exponentiation(^), and super-exponentiation(↑↑).

Binary operations are often written using infix notation such as $a * b$, $a + b$, $a \cdot b$ or (by juxtaposition with no symbol) ab rather than by functional notation of the form $f(a, b)$. Powers are usually also written without operator, but with the second argument as superscript.

Binary operations sometimes use prefix or postfix notation; this dispenses with parentheses. Prefix notation is also called Polish notation; postfix notation, also called reverse Polish notation, is probably more often encountered.

Pair and tuple

A binary operation, ab, depends on the ordered pair (a, b) and so $(ab)c$ (where the parentheses here mean first operate on the ordered pair (a, b) and then operate on the result of that using the ordered pair ((ab), c)) depends in general on the ordered pair $((a,b),c)$. Thus, for the general, non-associative case, binary operations can be represented with binary trees.

However:

- If the operation is associative, $(ab)c=a(bc)$, then the value depends only on the tuple (a,b,c).
- If the operation is commutative, $ab=ba$, then the value depends only on the multiset $\{ \{a,b\},c\}$.
- If the operation is both associative and commutative then the value depends only on the multiset $\{a,b,c\}$.
- If the operation is both associative and commutative and idempotent, $aa=a$, then the value depends only on the set $\{a,b,c\}$.

External binary operations

An **external binary operation** is a binary function from $K \times S$ to S. This differs from a binary operation in the strict sense in that K need not be S; its elements come from *outside*.

An example of an external binary operation is scalar multiplication in linear algebra. Here K is a field and S is a vector space over that field.

An external binary operation may alternatively be viewed as an action; K is acting on S.

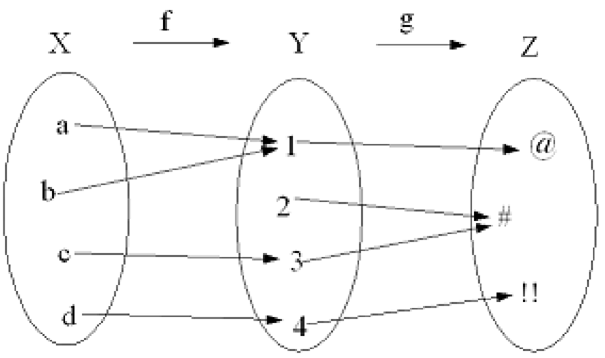
Note that the dot product of two vectors is not a binary operation, external or otherwise, as it maps from $S \times S$ to K, where K is a field and S is a vector space over K.

See also

- Iterated binary operation
- Unary operation
- Ternary operation

Function composition

In mathematics, **function composition** is the application of one function to the results of another. For instance, the functions f: $X \to Y$ and g: $Y \to Z$ can be *composed* by computing the output of g when it has an argument of $f(x)$ instead of x. Intuitively, if z is a function g of y and y is a function f of x, then z is a function of x.



$g \circ f$, the **composition** of f and g. For example, $(g \circ f)(c) = \#$.

Thus one obtains a function $g \circ f$: $X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ for all x in X. The notation $g \circ f$ is read as "g circle f", or "g composed with f", "g after f", "g following f", or just "g of f".

The composition of functions is always associative. That is, if f, g, and h are three functions with suitably chosen domains and codomains, then $f \circ (g \circ h) = (f \circ g) \circ h$, where the parentheses serve to indicate that composition is to be performed first for the parenthesized functions. Since there is no distinction between the choices of placement of parentheses, they may be safely left off.

The functions g and f are said to commute with each other if $g \circ f = f \circ g$. In general, composition of functions will not be commutative. Commutativity is a special property, attained only by particular functions, and often in special circumstances. For example, $|x| + 3 = |x + 3|$ only when $x \geq 0$. But a function always commutes with its inverse to produce the identity mapping.

Considering functions as special cases of relations (namely functional relations), one can analogously define composition of relations, which gives the formula for $g \circ f \subseteq X \times Z$ in terms of $f \subseteq X \times Y$ and $g \subseteq Y \times Z$.

Derivatives of compositions involving differentiable functions can be found using the chain rule. Higher derivatives of such functions are given by Faà di Bruno's formula.

The structures given by composition are axiomatized and generalized in category theory.

Example

As an example, suppose that an airplane's elevation at time t is given by the function $h(t)$ and that the oxygen concentration at elevation x is given by the function $c(x)$. Then $(c \circ h)(t)$ describes the oxygen concentration around the plane at time t.

Functional powers

If $Y \subseteq X$ then $f\colon X \to Y$ may compose with itself; this is sometimes denoted f^2. Thus:

$$(f \circ f)(x) = f(f(x)) = f^2(x)$$

$$(f \circ f \circ f)(x) = f(f(f(x))) = f^3(x)$$

Repeated composition of a function with itself is called **function iteration**.

The **functional powers** $f \circ f^n = f^n \circ f = f^{n+1}$ for natural n follow immediately.

- By convention, $f^0 = id_{D(f)}$ (the identity map on the domain of f).
- If $f\colon X \to X$ admits an inverse function, negative functional powers f^{-k} $(k > 0)$ are defined as the opposite power of the inverse function, $(f^{-1})^k$.

Note: If f takes its values in a ring (in particular for real or complex-valued f), there is a risk of confusion, as f^n could also stand for the n-fold product of f, e.g. $f^2(x) = f(x) \cdot f(x)$.

(For trigonometric functions, usually the latter is meant, at least for positive exponents. For example, in trigonometry, this superscript notation represents standard exponentiation when used with trigonometric functions: $\sin^2(x) = \sin(x) \cdot \sin(x)$. However, for negative exponents (especially −1), it nevertheless usually refers to the inverse function, e.g., $\tan^{-1} = \arctan$ ($\neq 1/\tan$).

In some cases, an expression for f in $g(x) = f^r(x)$ can be derived from the rule for g given non-integer values of r. This is called fractional iteration. For instance, a half iterate of a function f is a function g satisfying $g(g(x)) = f(x)$. Another example would be that where f is the successor function, $f^r(x) = x + r$. This idea can be generalized so that the iteration count becomes a continuous parameter; in this case, such a system is called a flow.

Iterated functions and flows occur naturally in the study of fractals and dynamical systems.

Composition monoids

Suppose one has two (or more) functions $f\colon X \to X$, $g\colon X \to X$ having the same domain and codomain. Then one can form long, potentially complicated chains of these functions composed together, such as $f \circ f \circ g \circ f$. Such long chains have the algebraic structure of a monoid, called transformation monoid or composition monoid. In general, composition monoids can have remarkably complicated structure. One particular notable example is the de Rham curve. The set of *all* functions $f\colon X \to X$ is called the full transformation semigroup on X.

If the functions are bijective, then the set of all possible combinations of these functions forms a transformation group; and one says that the group is generated by these functions.

The set of all bijective functions $f\colon X \to X$ form a group with respect to the composition operator. This is the symmetric group, also sometimes called the **composition group**.

Alternative notations

- Many mathematicians omit the composition symbol, writing gf for $g \circ f$.
- In the mid-20th century, some mathematicians decided that writing "$g \circ f$" to mean "first apply f, then apply g" was too confusing and decided to change notations. They write "xf" for "$f(x)$" and "$(xf)g$" for "$g(f(x))$". This can be more natural and seem simpler than writing functions on the left in some areas – in linear algebra, for instance, where x is a row vector and f and g denote matrices and the composition is by matrix multiplication. This alternative notation is called postfix notation. The order is important because matrix multiplication is non-commutative. Successive transformations applying and composing to the right agrees with the left-to-right reading sequence.
- Mathematicians who use postfix notation may write "fg", meaning first do f then do g, in keeping with the order the symbols occur in postfix notation, thus making the notation "fg" ambiguous. Computer scientists may write "$f;g$" for this, thereby disambiguating the order of composition. To distinguish the left composition operator from a text semicolon, in the Z notation a fat semicolon ⨟ (U+2A1F) is used for left relation composition. Since all functions are binary relations, it is correct to use the fat semicolon for function composition as well (see the article on Composition of relations for further details on this notation).

Composition operator

Given a function g, the **composition operator** C_g is defined as that operator which maps functions to functions as

$$C_g f = f \circ g.$$

Composition operators are studied in the field of operator theory.

See also

- Combinatory logic
- Composition of relations, the generalization to relations
- Function composition (computer science)
- Functional decomposition
- Higher-order function
- Lambda calculus

External links

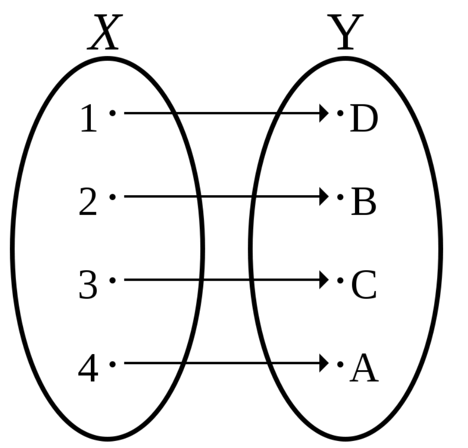
- "Composition of Functions [1]" by Bruce Atwood, the Wolfram Demonstrations Project, 2007.

References

[1] http://demonstrations.wolfram.com/CompositionOfFunctions/

Bijection

In mathematics, a **bijection**, or a **bijective function** is a function f from a set X to a set Y with the property that, for every y in Y, there is exactly one x in X such that $f(x) = y$ and no unmapped element exists in either X or Y.



A bijective function.

Alternatively, f is bijective if it is a **one-to-one correspondence** between those sets; i.e., both **one-to-one** (injective) and **onto** (surjective). (*One-to-one function* means *one-to-one correspondence* (i.e., *bijection*) to some authors, but *injection* to others.)

For example, consider the function *succ*, defined from the set of integers $\mathbb{Z}$ to $\mathbb{Z}$, that to each integer x associates the integer succ(x) = x + 1. For another example, consider the function sumdif that to each pair (x,y) of real numbers associates the pair sumdif$(x,y) = (x + y, x - y)$.

A bijective function from a set to itself is also called a **permutation**.

The set of all bijections from X to Y is denoted as $X \leftrightarrow Y$. (Sometimes this notation is reserved for binary relations, and bijections are denoted by X ⤖ Y instead.) Occasionally, the set of permutations of a single set X may be denoted X!.

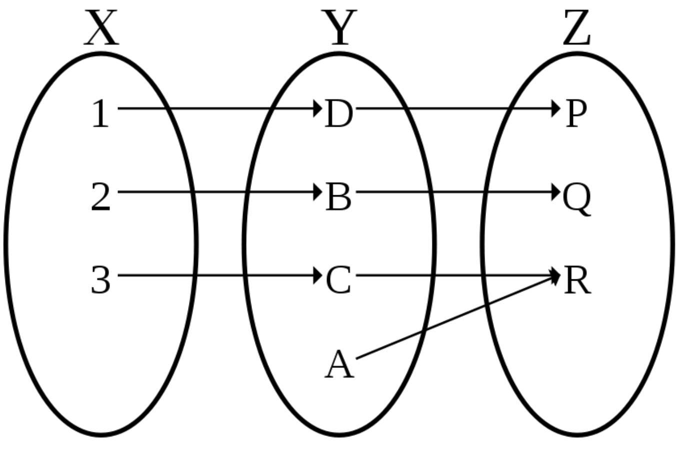
Bijective functions play a fundamental role in many areas of mathematics, for instance in the definition of isomorphism (and related concepts such as homeomorphism and diffeomorphism), permutation group, projective map, and many others.

Composition and inverses

A function f is bijective if and only if its inverse relation f^{-1} is a function. In that case, f^{-1} is also a bijection.

The composition $g \circ f$ of two bijections f: $X \leftrightarrow Y$ and g: $Y \leftrightarrow Z$ is a bijection. The inverse of $g \circ f$ is $(g \circ f)^{-1} = (f^{-1}) \circ (g^{-1})$.

On the other hand, if the composition $g \circ f$ of two functions is bijective, we can only say that f is injective and g is surjective.



A bijection composed of an injection (left) and a surjection (right).

A relation f from X to Y is a bijective function if and only if there exists another relation g from Y to X such that $g \circ f$ is the identity function on X, and $f \circ g$ is the identity function on Y. Consequently, the sets have the same cardinality.

Bijections and cardinality

If X and Y are finite sets, then there exists a bijection between the two sets X and Y iff X and Y have the same number of elements. Indeed, in axiomatic set theory, this is taken as the very *definition* of "same number of elements", and generalising this definition to infinite sets leads to the concept of cardinal number, a way to distinguish the various sizes of infinite sets.

Examples and counterexamples

- For any set X, the identity function id_X from X to X, defined by $\mathrm{id}_X(x) = x$, is bijective.
- The function f from the real line **R** to **R** defined by $f(x) = 2x + 1$ is bijective, since for each y there is a unique $x = (y - 1)/2$ such that $f(x) = y$.
- The exponential function $g : \mathbf{R} \longrightarrow \mathbf{R}$, with $g(x) = e^x$, is not bijective: for instance, there is no x in **R** such that $g(x) = -1$, showing that g is not surjective. However if the codomain is changed to be the positive real numbers $\mathbf{R}^+ = (0,+\infty)$, then g becomes bijective; its inverse is the natural logarithm function ln.
- The function $h : \mathbf{R} \to [0,+\infty)$ with $h(x) = x^2$ is not bijective: for instance, $h(-1) = h(+1) = 1$, showing that h is not injective. However, if the domain too is changed to $[0,+\infty)$, then h becomes bijective; its inverse is the positive square root function.
- $\mathbb{R} \to \mathbb{R} : x \mapsto (x-1)x(x+1) = x^3 - x$ is not a bijection because −1, 0, and +1 are all in the domain and all map to 0.
- $\mathbb{R} \to [-1,1] : x \mapsto \sin(x)$ is not a bijection because π/3 and 2π/3 are both in the domain and both map to (√3)/2.

Properties

- A function f from the real line **R** to **R** is bijective if and only if its plot is intersected by any horizontal or vertical line at exactly one point.
- If X is a set, then the bijective functions from X to itself, together with the operation of functional composition (∘), form a group, the symmetric group of X, which is denoted variously by $S(X)$, S_X, or $X!$ (the last reads "X factorial").
- For a subset A of the domain with cardinality $|A|$ and subset B of the codomain with cardinality $|B|$, one has the following equalities:

  $|f(A)| = |A|$ and $|f^{-1}(B)| = |B|$.
- If X and Y are finite sets with the same cardinality, and $f: X \to Y$, then the following are equivalent:
  1. f is a bijection.
  2. f is a surjection.
  3. f is an injection.
- At least for a finite set S, there is a bijection between the set of possible total orderings of the elements and the set of bijections from S to S. That is to say, the number of permutations of elements of S is the same as the number of total orderings of that set -- namely, $n!$.

Bijections and category theory

Formally, bijections are precisely the isomorphisms in the category **Set** of sets and functions. However, the bijections are not always the isomorphisms. For example, in the category **Top** of topological spaces and continuous functions, the isomorphisms must be homeomorphisms in addition to being bijections.

See also

- Category theory
- Injective function
- Symmetric group
- Surjective function
- Bijective numeration
- Bijective proof

External links

- Earliest Uses of Some of the Words of Mathematics: entry on Injection, Surjection and Bijection has the history of Injection and related terms. [1]

References

[1] http://jeff560.tripod.com/i.html

Associativity

In mathematics, **associativity** is a property that a binary operation can have. It means that, within an expression containing two or more occurrences in a row of the same associative operator, the order in which the operations are performed does not matter as long as the sequence of the operands is not changed. That is, rearranging the parentheses in such an expression will not change its value. Consider for instance the equation

$$(5+2)+1=5+(2+1)=8$$

Even though the parentheses were rearranged (the left side requires adding 5 and 2 first, then adding 1 to the result, whereas the right side requires adding 2 and 1 first, then 5), the value of the expression was not altered. Since this holds true when performing addition on any real numbers, we say that "addition of real numbers is an associative operation."

Associativity is not to be confused with commutativity. Commutativity justifies changing the order or sequence of the operands within an expression while associativity does not. For example,

$$(5+2)+1=5+(2+1)$$

is an example of associativity because the parentheses were changed (and consequently the order of operations during evaluation) while the operands 5, 2, and 1 appeared in the exact same order from left to right in the expression.

$$(5+2)+1=(2+5)+1$$

is not an example of associativity because the operand sequence changed when the 2 and 5 switched places.

Associative operations are abundant in mathematics; in fact, many algebraic structures (such as semigroups and categories) explicitly require their binary operations to be associative.

However, many important and interesting operations are non-associative; one common example would be the vector cross product.

Definition

Formally, a binary operation $*$ on a set *S* is called **associative** if it satisfies the **associative law**:

$$(x * y) * z = x * (y * z) \qquad \text{for all } x, y, z \in S.$$

*Using * to denote a binary operation performed on a set*

$$(xy)z = x(yz) = xyz \qquad \text{for all } x, y, z \in S.$$

An example of multiplicative associativity

The evaluation order does not affect the value of such expressions, and it can be shown that the same holds for expressions containing *any* number of $*$ operations. Thus, when $*$ is associative, the evaluation order can therefore be left unspecified without causing ambiguity, by omitting the parentheses and writing simply:

$$xyz,$$

However, it is important to remember that changing the order of operations does not involve or permit moving the operands around within the expression; the sequence of operands is always unchanged.

A very different perspective is obtained by rephrasing associativity using functional notation: $f(f(x,y),z) = f(x,f(y,z))$: when expressed in this form, associativity becomes less obvious.

Associativity can be generalized to n-ary operations. Ternary associativity is (abc)de = a(bcd)e = ab(cde), i.e. the string abcde with any three adjacent elements bracketed. N-ary associativity is a string of length n+(n-1) with any n adjacent elements bracketed[1] .

Examples

Some examples of associative operations include the following.

- The prototypical example of an associative operation is string concatenation: the concatenation of "hello", ", ", "world" can be computed by concatenating the first two strings (giving "hello, ") and appending the third string ("world"), or by joining the second and third string (giving ", world") and concatenating the first string ("hello") with the result.
- In arithmetic, addition and multiplication of real numbers are associative; i.e.,

$$\left.\begin{array}{l}(x+y)+z = x+(y+z) = x+y+z\\ (x\,y)z = x(y\,z) = x\,y\,z\end{array}\right\}\text{for all } x, y, z \in \mathbb{R}.$$

- Addition and multiplication of complex numbers and quaternions is associative. Addition of octonions is also associative, but multiplication of octonions is non-associative.
- The greatest common divisor and least common multiple functions act associatively.

$$\left.\begin{array}{l}\gcd(\gcd(x,y),z) = \gcd(x,\gcd(y,z)) = \gcd(x,y,z)\\ \operatorname{lcm}(\operatorname{lcm}(x,y),z) = \operatorname{lcm}(x,\operatorname{lcm}(y,z)) = \operatorname{lcm}(x,y,z)\end{array}\right\}\text{ for all } x, y, z \in \mathbb{Z}.$$

- Because linear transformations are functions that can be represented by matrices with matrix multiplication being the representation of functional composition, one can immediately conclude that matrix multiplication is associative.
- Taking the intersection or the union of sets:

$$\left.\begin{array}{l}(A\cap B)\cap C = A\cap(B\cap C) = A\cap B\cap C\\ (A\cup B)\cup C = A\cup(B\cup C) = A\cup B\cup C\end{array}\right\}\text{for all sets } A, B, C.$$

- If *M* is some set and *S* denotes the set of all functions from *M* to *M*, then the operation of functional composition on *S* is associative:

$$(f\circ g)\circ h = f\circ(g\circ h) = f\circ g\circ h \qquad \text{for all } f, g, h \in S.$$

- Slightly more generally, given four sets *M*, *N*, *P* and *Q*, with *h*: *M* to *N*, *g*: *N* to *P*, and *f*: *P* to *Q*, then

$$(f \circ g) \circ h = f \circ (g \circ h) = f \circ g \circ h$$

as before. In short, composition of maps is always associative.

- Consider a set with three elements, A, B, and C. The following operation:

+			
×	**A**	**B**	**C**
A	A	A	A
B	A	B	C
C	A	A	A

is associative. Thus, for example, A(BC)=(AB)C. This mapping is not commutative.

Non-associativity

A binary operation $*$ on a set *S* that does not satisfy the associative law is called **non-associative**. Symbolically,

$$(x * y) * z \neq x * (y * z) \qquad \text{for some } x, y, z \in S.$$

For such an operation the order of evaluation *does* matter. For example:

- Subtraction

$$(5 - 3) - 2 \neq 5 - (3 - 2)$$

- division

$$(4/2)/2 \neq 4/(2/2)$$

- Exponentiation

$$2^{(1^2)} \neq (2^1)^2$$

Also note that infinite sums are not generally associative, for example:

$$(1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0$$

whereas

$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + \ldots = 1$$

The study of non-associative structures arises from reasons somewhat different from the mainstream of classical algebra. One area within non-associative algebra that has grown very large is that of Lie algebras. There the associative law is replaced by the Jacobi identity. Lie algebras abstract the essential nature of infinitesimal transformations, and have become ubiquitous in mathematics. They are an example of non-associative algebras.

There are other specific types of non-associative structures that have been studied in depth. They tend to come from some specific applications. Some of these arise in combinatorial mathematics. Other examples: Quasigroup, Quasifield, Nonassociative ring.

Notation for non-associative operations

In general, parentheses must be used to indicate the order of evaluation if a non-associative operation appears more than once in an expression. However, mathematicians agree on a particular order of evaluation for several common non-associative operations. This is simply a notational convention to avoid parentheses.

A **left-associative** operation is a non-associative operation that is conventionally evaluated from left to right, i.e.,

$$\left.\begin{array}{l} x*y*z=(x*y)*z \\ w*x*y*z=((w*x)*y)*z \\ \text{etc.} \end{array}\right\} \text{for all } w,x,y,z\in S$$

while a **right-associative** operation is conventionally evaluated from right to left:

$$\left.\begin{array}{l} x*y*z=x*(y*z) \\ w*x*y*z=w*(x*(y*z)) \\ \text{etc.} \end{array}\right\} \text{for all } w,x,y,z\in S$$

Both left-associative and right-associative operations occur. Left-associative operations include the following:

- Subtraction and division of real numbers:

$$x-y-z=(x-y)-z \qquad \text{for all } x,y,z\in\mathbb{R};$$
$$x/y/z=(x/y)/z \qquad \text{for all } x,y,z\in\mathbb{R} \text{ with } y\neq 0, z\neq 0.$$

- Function application:

$$(f\,x\,y)=((f\,x)\,y)$$

  This notation can be motivated by the currying isomorphism.

Right-associative operations include the following:

- Exponentiation of real numbers:

$$x^{y^z}=x^{(y^z)}.$$

  The reason exponentiation is right-associative is that a repeated left-associative exponentiation operation would be less useful. Multiple appearances could (and would) be rewritten with multiplication:

$$(x^y)^z=x^{(yz)}.$$

- Function definition

$$\mathbb{Z}\to\mathbb{Z}\to\mathbb{Z}=\mathbb{Z}\to(\mathbb{Z}\to\mathbb{Z})$$
$$x\mapsto y\mapsto x-y=x\mapsto(y\mapsto x-y)$$

  Using right-associative notation for these operations can be motivated by the Curry-Howard correspondence and by the currying isomorphism.

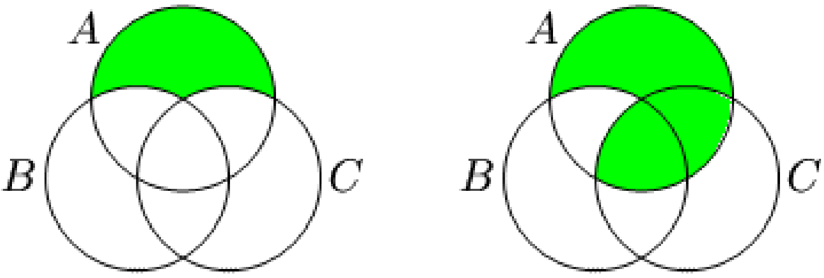
Non-associative operations for which no conventional evaluation order is defined include the following.

- Taking the pairwise average of real numbers:

$$\frac{(x+y)/2+z}{2}\neq\frac{x+(y+z)/2}{2} \qquad \text{for all } x,y,z\in\mathbb{R} \text{ with } x\neq z.$$

- Taking the relative complement of sets:

$$(A\backslash B)\backslash C\neq A\backslash(B\backslash C) \qquad \text{for some sets } A,B,C.$$



The green part in the left Venn diagram represents $(A \backslash B) \backslash C$. The green part in the right Venn diagram represents $A \backslash (B \backslash C)$.

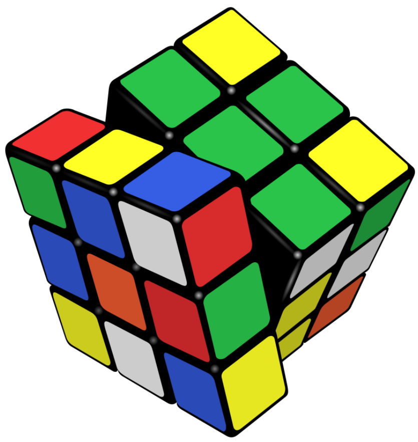
See also

- Light's associativity test
- A semigroup is a set with a closed associative binary operation.
- Commutativity and distributivity are two other frequently discussed properties of binary operations.
- Power associativity and alternativity are weak forms of associativity.

References

[1] Dudek, W.A. (2001), "On some old problems in n-ary groups" (http://www.quasigroups.eu/contents/contents8.php?m=trzeci), *Quasigroups and Related Systems* **8**: 15–36, .

Group (mathematics)



The possible manipulations of this Rubik's Cube form a group.

In mathematics, a **group** is an algebraic structure consisting of a set together with an operation that combines any two of its elements to form a third element. To qualify as a group, the set and the operation must satisfy a few conditions called group axioms, namely closure, associativity, identity and invertibility. While these are familiar from many mathematical structures, such as number systems—for example, the integers endowed with the addition operation form a group—the formulation of the axioms is detached from the concrete nature of the group and its operation. This allows one to handle entities of very different mathematical origins in a flexible way, while retaining essential structural aspects of many objects in abstract algebra and beyond. The ubiquity of groups in numerous areas—both within and outside mathematics—makes them a central organizing principle of contemporary mathematics.[1] [2]

Groups share a fundamental kinship with the notion of symmetry. A symmetry group encodes symmetry features of a geometrical object: it consists of the set of transformations that leave the object unchanged, and the operation of combining two such transformations by performing one after the other. Such symmetry groups, particularly the continuous Lie groups, play an important role in many academic disciplines. Matrix groups, for example, can be used to understand fundamental physical laws underlying special relativity and symmetry phenomena in molecular chemistry.

The concept of a group arose from the study of polynomial equations, starting with Évariste Galois in the 1830s. After contributions from other fields such as number theory and geometry, the group notion was generalized and firmly established around 1870. Modern group theory—a very active mathematical discipline—studies groups in their own right.a[›] To explore groups, mathematicians have devised various notions to break groups into smaller, better-understandable pieces, such as subgroups, quotient groups and simple groups. In addition to their abstract properties, group theorists also study the different ways in which a group can be expressed concretely (its group representations), both from a theoretical and a computational point of view. A particularly rich theory has been developed for finite groups, which culminated with the monumental classification of finite simple groups completed in 1983. Since the mid-1980s, geometric group theory, which studies finitely generated groups as geometric objects, has become a particularly active area in group theory.

Definition and illustration

First example: the integers

One of the most familiar groups is the set of integers **Z** which consists of the numbers

..., −4, −3, −2, −1, 0, 1, 2, 3, 4, ...[3]

The following properties of integer addition serve as a model for the abstract group axioms given in the definition below.

1. For any two integers a and b, the sum $a + b$ is also an integer. In other words, the process of adding integers two at a time can never yield a result that is not an integer. This property is known as *closure* under addition.
2. For all integers a, b and c, $(a + b) + c = a + (b + c)$. Expressed in words, adding a to b first, and then adding the result to c gives the same final result as adding a to the sum of b and c, a property known as *associativity*.
3. If a is any integer, then $0 + a = a + 0 = a$. Zero is called the *identity element* of addition because adding it to any integer returns the same integer.
4. For every integer a, there is an integer b such that $a + b = b + a = 0$. The integer b is called the *inverse element* of the integer a and is denoted $-a$.

Definition

The integers, together with the operation "+", form a mathematical object belonging to a broad class sharing similar structural aspects. To appropriately understand these structures without dealing with every concrete case separately, the following abstract definition is developed to encompass the above example along with many others, one of which is the symmetry group detailed below.

A group is a set, G, together with an operation "•" that combines any two elements a and b to form another element denoted $a \bullet b$. The symbol "•" is a general placeholder for a concretely given operation, such as the addition above. To qualify as a group, the set and operation, $(G, \bullet)$, must satisfy four requirements known as the *group axioms*:[4]

Closure

For all a, b in G, the result of the operation $a \bullet b$ is also in G.b[›]

Associativity

For all a, b and c in G, the equation $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ holds.

Identity element

There exists an element e in G, such that for every element a in G, the equation $e \bullet a = a \bullet e = a$ holds.

Inverse element

For each a in G, there exists an element b in G such that $a \bullet b = b \bullet a = e$, where e is the identity element.

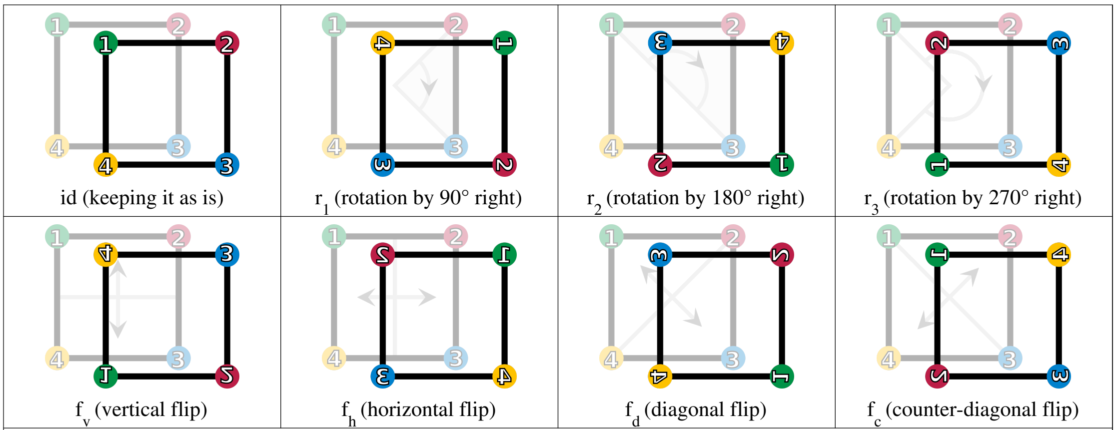
The order in which the group operation is carried out can be significant. In other words, the result of combining element a with element b need not yield the same result as combining element b with element a; the equation

$a \bullet b = b \bullet a$

may not always be true. This equation does always hold in the group of integers under addition, because $a + b = b + a$ for any two integers (commutativity of addition). However, it does not always hold in the symmetry group below. Groups for which the equation $a \bullet b = b \bullet a$ always holds are called *abelian* (in honor of Niels Abel). Thus, the integer addition group is abelian, but the following symmetry group is not.

Second example: a symmetry group

The symmetries (i.e., rotations and reflections) of a square form a group called a dihedral group, and denoted D_4.[5] The following symmetries occur:



id (keeping it as is) | r_1 (rotation by 90° right) | r_2 (rotation by 180° right) | r_3 (rotation by 270° right)

f_v (vertical flip) | f_h (horizontal flip) | f_d (diagonal flip) | f_c (counter-diagonal flip)

The elements of the symmetry group of the square (D_4). The vertices are colored and numbered only to visualize the operations.

- the identity operation leaving everything unchanged, denoted id;
- rotations of the square by 90° right, 180° right, and 270° right, denoted by r_1, r_2 and r_3, respectively;
- reflections about the vertical and horizontal middle line (f_h and f_v), or through the two diagonals (f_d and f_c).

Any two symmetries a and b can be composed; i.e., applied one after another. The result of performing first a and then b is written symbolically *from right to left* as

> $b \bullet a$ ("apply the symmetry b after performing the symmetry a". The right-to-left notation stems from composition of functions).

The group table on the right lists the results of all such compositions possible. For example, rotating by 270° right (r_3) and then flipping horizontally (f_h) is the same as performing a reflection along the diagonal (f_d). Using the above symbols, highlighted in blue in the group table:

$$f_h \bullet r_3 = f_d.$$

Group table of D_4

•	**id**	$\mathbf{r_1}$	$\mathbf{r_2}$	$\mathbf{r_3}$	$\mathbf{f_v}$	$\mathbf{f_h}$	$\mathbf{f_d}$	$\mathbf{f_c}$
id	id	r_1	r_2	r_3	f_v	f_h	f_d	f_c
$\mathbf{r_1}$	r_1	r_2	r_3	id	f_c	f_d	f_v	f_h
$\mathbf{r_2}$	r_2	r_3	id	r_1	f_h	f_v	f_c	f_d
$\mathbf{r_3}$	r_3	id	r_1	r_2	f_d	f_c	f_h	f_v
$\mathbf{f_v}$	f_v	f_d	f_h	f_c	id	r_2	r_1	r_3
$\mathbf{f_h}$	f_h	f_c	f_v	f_d	r_2	id	r_3	r_1
$\mathbf{f_d}$	f_d	f_h	f_c	f_v	r_3	r_1	id	r_2
$\mathbf{f_c}$	f_c	f_v	f_d	f_h	r_1	r_3	r_2	id

The elements id, r_1, r_2, and r_3 form a subgroup, highlighted in red (upper left region). A left and right coset of this subgroup is highlighted in green (in the last row) and yellow (last column), respectively.

Given this set of symmetries and the described operation, the group axioms can be understood as follows:

1. The closure axiom demands that the composition $b \bullet a$ of any two symmetries a and b is also a symmetry. Another example for the group operation is

   $r_3 \bullet f_h = f_c$,

   i.e. rotating 270° right after flipping horizontally equals flipping along the counter-diagonal (f_c). Indeed every other combination of two symmetries still gives a symmetry, as can be checked using the group table.
2. The associativity constraint deals with composing more than two symmetries: given three elements a, b and c of D_4, there are two possible ways of computing "a then b then c". The requirement

   $(a \bullet b) \bullet c = a \bullet (b \bullet c)$

   means that the composition of the three elements is independent of the priority of the operations, i.e. composing first a after b, and c to the result $a \bullet b$ thereof amounts to performing a after the composition of b and c. For example, $(f_d \bullet f_v) \bullet r_2 = f_d \bullet (f_v \bullet r_2)$ can be checked using the group table at the right

   $$(f_d \bullet f_v) \bullet r_2 = r_3 \bullet r_2 = r_1, \text{ which equals}$$
   $$f_d \bullet (f_v \bullet r_2) = f_d \bullet f_h = r_1.$$

3. The identity element is the symmetry id leaving everything unchanged: for any symmetry a, performing id after a (or a after id) equals a, in symbolic form,

   $\mathrm{id} \bullet a = a$,

   $a \bullet \mathrm{id} = a$.
4. An inverse element undoes the transformation of some other element. Every symmetry can be undone: each of transformations—identity id, the flips f_h, f_v, f_d, f_c and the 180° rotation r_2—is its own inverse, because performing each one twice brings the square back to its original orientation. The rotations r_3 and r_1 are each other's inverse, because rotating one way and then by the same angle the other way leaves the square unchanged. In symbols,

   $f_h \bullet f_h = \mathrm{id}$,

   $r_3 \bullet r_1 = r_1 \bullet r_3 = \mathrm{id}$.

In contrast to the group of integers above, where the order of the operation is irrelevant, it does matter in D_4: $f_h \bullet r_1 = f_c$ but $r_1 \bullet f_h = f_d$. In other words, D_4 is not abelian, which makes the group structure more difficult than the integers introduced first.

History

The modern concept of an abstract group developed out of several fields of mathematics.[6] [7] [8] The original motivation for group theory was the quest for solutions of polynomial equations of degree higher than 4. The 19th-century French mathematician Évariste Galois, extending prior work of Paolo Ruffini and Joseph-Louis Lagrange, gave a criterion for the solvability of a particular polynomial equation in terms of the symmetry group of its roots (solutions). The elements of such a Galois group correspond to certain permutations of the roots. At first, Galois' ideas were rejected by his contemporaries, and published only posthumously.[9] [10] More general permutation groups were investigated in particular by Augustin Louis Cauchy. Arthur Cayley's *On the theory of groups, as depending on the symbolic equation* $\theta^n = 1$ (1854) gives the first abstract definition of a finite group.[11]

Geometry was a second field in which groups were used systematically, especially symmetry groups as part of Felix Klein's 1872 Erlangen program.[12] After novel geometries such as hyperbolic and projective geometry had emerged, Klein used group theory to organize them in a more coherent way. Further advancing these ideas, Sophus Lie founded the study of Lie groups in 1884.[13]

The third field contributing to group theory was number theory. Certain abelian group structures had been used implicitly in Carl Friedrich Gauss' number-theoretical work *Disquisitiones Arithmeticae* (1798), and more explicitly by Leopold Kronecker.[14] In 1847, Ernst Kummer led early attempts to prove Fermat's Last Theorem to a climax by developing groups describing factorization into prime numbers.[15]

The convergence of these various sources into a uniform theory of groups started with Camille Jordan's *Traité des substitutions et des équations algébriques* (1870).[16] Walther von Dyck (1882) gave the first statement of the modern definition of an abstract group.[17] As of the 20th century, groups gained wide recognition by the pioneering work of Ferdinand Georg Frobenius and William Burnside, who worked on representation theory of finite groups, Richard Brauer's modular representation theory and Issai Schur's papers.[18] The theory of Lie groups, and more generally locally compact groups was pushed by Hermann Weyl, Élie Cartan and many others.[19] Its algebraic counterpart, the theory of algebraic groups, was first shaped by Claude Chevalley (from the late 1930s) and later by pivotal work of Armand Borel and Jacques Tits.[20]

The University of Chicago's 1960–61 Group Theory Year brought together group theorists such as Daniel Gorenstein, John G. Thompson and Walter Feit, laying the foundation of a collaboration that, with input from numerous other mathematicians, classified all finite simple groups in 1982. This project exceeded previous mathematical endeavours by its sheer size, in both length of proof and number of researchers. Research is ongoing to simplify the proof of this classification.[21] These days, group theory is still a highly active mathematical branch crucially impacting many other fields.[a][›]

Elementary consequences of the group axioms

Basic facts about all groups that can be obtained directly from the group axioms are commonly subsumed under *elementary group theory*.[22] For example, repeated applications of the associativity axiom show that the unambiguity of

$$a \bullet b \bullet c = (a \bullet b) \bullet c = a \bullet (b \bullet c)$$

generalizes to more than three factors. Because this implies that parentheses can be inserted anywhere within such a series of terms, parentheses are usually omitted.[23]

The axioms may be weakened to assert only the existence of a left identity and left inverses. Both can be shown to be actually two-sided, so the resulting definition is equivalent to the one given above.[24]

Uniqueness of identity element and inverses

Two important consequences of the group axioms are the uniqueness of the identity element and the uniqueness of inverse elements. There can be only one identity element in a group, and each element in a group has exactly one inverse element. Thus, it is customary to speak of *the* identity, and *the* inverse of an element.[25]

To prove the uniqueness of an inverse element of a, suppose that a has two inverses, denoted l and r. Then

l	$=$	$l \bullet e$	as e is the identity element
	$=$	$l \bullet (a \bullet r)$	because r is an inverse of a, so $e = a \bullet r$
	$=$	$(l \bullet a) \bullet r$	by associativity, which allows to rearrange the parentheses
	$=$	$e \bullet r$	since l is an inverse of a, i.e. $l \bullet a = e$
	$=$	r	for e is the identity element

Hence the two extremal terms l and r are connected by a chain of equalities, so they agree. In other words there is only one inverse element of a.

Division

In groups, it is possible to perform division: given elements a and b of the group G, there is exactly one solution x in G to the equation $x \bullet a = b$.[25] In fact, right multiplication of the equation by a^{-1} gives the solution $x = x \bullet a \bullet a^{-1} = b \bullet a^{-1}$. Similarly there is exactly one solution y in G to the equation $a \bullet y = b$, namely $y = a^{-1} \bullet b$. In general, x and y need not agree.

Basic concepts

To understand groups beyond the level of mere symbolic manipulations as above, more structural concepts have to be employed.[c][›] There is a conceptual principle underlying all of the following notions: to take advantage of the structure offered by groups (which sets, being "structureless", don't have), constructions related to groups have to be *compatible* with the group operation. This compatibility manifests itself in the following notions in various ways. For example, groups can be related to each other via functions called group homomorphisms. By the mentioned principle, they are required to respect the group structures in a precise sense. The structure of groups can also be understood by breaking them into pieces called subgroups and quotient groups. The principle of "preserving structures"—a recurring topic in mathematics throughout—is an instance of working in a category, in this case the category of groups.[26]

Group homomorphisms

Group homomorphisms[g][›] are functions that preserve group structure. A function a: $G \to H$ between two groups is a homomorphism if the equation

$$a(g \bullet k) = a(g) \bullet a(k).$$

holds for all elements g, k in G, i.e. the result is the same when performing the group operation after or before applying the map a. This requirement ensures that $a(1_G) = 1_H$, and also $a(g)^{-1} = a(g^{-1})$ for all g in G. Thus a group homomorphism respects all the structure of G provided by the group axioms.[27]

Two groups G and H are called isomorphic if there exist group homomorphisms a: $G \to H$ and b: $H \to G$, such that applying the two functions one after another (in each of the two possible orders) equal the identity function of G and H, respectively. That is, $a(b(h)) = h$ and $b(a(g)) = g$ for any g in G and h in H. From an abstract point of view, isomorphic groups carry the same information. For example, proving that $g \bullet g = 1$ for some element g of G is equivalent to proving that $a(g) \bullet a(g) = 1$, because applying a to the first equality yields the second, and applying b to the second gives back the first.

Subgroups

Informally, a *subgroup* is a group H contained within a bigger one, G.[28] Concretely, the identity element of G is contained in H, and whenever h_1 and h_2 are in H, then so are $h_1 \cdot h_2$ and h_1^{-1}, so the elements of H, equipped with the group operation on G restricted to H, form indeed a group.

In the example above, the identity and the rotations constitute a subgroup $R = \{\mathrm{id}, r_1, r_2, r_3\}$, highlighted in red in the group table above: any two rotations composed are still a rotation, and a rotation can be undone by (i.e. is inverse to) the complementary rotations 270° for 90°, 180° for 180°, and 90° for 270° (note that rotation in the opposite direction is not defined). The subgroup test is a necessary and sufficient condition for a subset H of a group G to be a subgroup: it is sufficient to check that $g^{-1}h \in H$ for all elements g, $h \in H$. Knowing the subgroups is important in understanding the group as a whole.[d]

Given any subset S of a group G, the subgroup generated by S consists of products of elements of S and their inverses. It is the smallest subgroup of G containing S.[29] In the introductory example above, the subgroup generated by r_2 and f_v consists of these two elements, the identity element id and $f_h = f_v \cdot r_2$. Again, this is a subgroup, because combining any two of these four elements or their inverses (which are, in this particular case, these same elements) yields an element of this subgroup.

Cosets

In many situations it is desirable to consider two group elements the same if they differ by an element of a given subgroup. For example, in D_4 above, once a flip is performed, the square never gets back to the r_2 configuration by just applying the rotation operations (and no further flips), i.e. the rotation operations are irrelevant to the question whether a flip has been performed. Cosets are used to formalize this insight: a subgroup H defines left and right cosets, which can be thought of as translations of H by arbitrary group elements g. In symbolic terms, the *left* and *right coset* of H containing g are

$gH = \{gh, h \in H\}$ and $Hg = \{hg, h \in H\}$, respectively.[30]

The cosets of any subgroup H form a partition of G; that is, the union of all left cosets is equal to G and two left cosets are either equal or have an empty intersection.[31] The first case $g_1H = g_2H$ happens precisely when $g_1^{-1}g_2 \in H$, i.e. if the two elements differ by an element of H. Similar considerations apply to the right cosets of H. The left and right cosets of H may or may not be equal. If they are, i.e. for all g in G, $gH = Hg$, then H is said to be a *normal subgroup*. One may then simply refer to N as the set of cosets.

In D_4, the introductory symmetry group, the left cosets gR of the subgroup R consisting of the rotations are either equal to R, if g is an element of R itself, or otherwise equal to $U = f_vR = \{f_v, f_d, f_h, f_c\}$ (highlighted in green). The subgroup R is also normal, because $f_vR = U = Rf_v$ and similarly for any element other than f_v.

Quotient groups

In addition to disregarding the internal structure of a subgroup by considering its cosets, it is desirable to endow this coarser entity with a group law called *quotient group* or *factor group*. For this to be possible, the subgroup has to be normal. Given any normal subgroup N, the quotient group is defined by

$G / N = \{gN, g \in G\}$, "G modulo N".[32]

This set inherits a group operation (sometimes called coset multiplication, or coset addition) from the original group G: $(gN) \cdot (hN) = (gh)N$ for all g and h in G. This definition is motivated by the idea (itself an instance of general structural considerations outlined above) that the map $G \to G / N$ that associates to any element g its coset gN be a group homomorphism, or by general abstract considerations called universal properties. The coset $eN = N$ serves as the identity in this group, and the inverse of gN in the quotient group is $(gN)^{-1} = (g^{-1})N$.[e]

•	**R**	**U**
R	*R*	*U*
U	*U*	*R*

Group table of the quotient group D_4 / *R*.

The elements of the quotient group D_4 / *R* are *R* itself, which represents the identity, and $U = f_v R$. The group operation on the quotient is shown at the right. For example, $U \bullet U = f_v R \bullet f_v R = (f_v \bullet f_v)R = R$. Both the subgroup $R = \{\mathrm{id}, r_1, r_2, r_3\}$, as well as the corresponding quotient are abelian, whereas D_4 is not abelian. Building bigger groups by smaller ones, such as D_4 from its subgroup *R* and the quotient D_4 / *R* is abstracted by a notion called semidirect product.

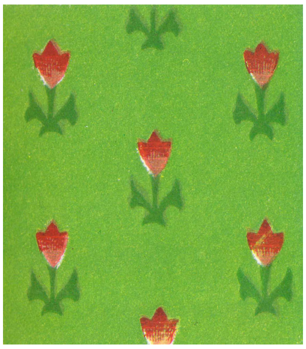
Quotient and subgroups together form a way of describing every group by its *presentation*: any group is the quotient of the free group over the *generators* of the group, quotiented by the subgroup of *relations*. The dihedral group D_4, for example, can be generated by two elements *r* and *f* (for example, $r = r_1$, the right rotation and $f = f_v$ the vertical (or any other) flip), which means that every symmetry of the square is a finite composition of these two symmetries or their inverses. Together with the relations

$$r^4 = f^2 = (rf)^2 = 1,$$[33]

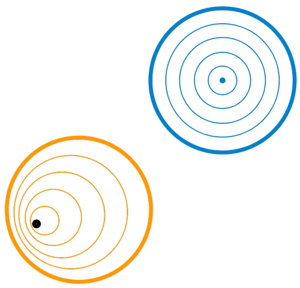
the group is completely described. A presentation of a group can also be used to construct the Cayley graph, a device used to graphically capture discrete groups.

Sub- and quotient groups are related in the following way: a subset *H* of *G* can be seen as an injective map $H \to G$, i.e. any element of the target has at most one element that maps to it. The counterpart to injective maps are surjective maps (every element of the target is mapped onto), such as the canonical map $G \to G / N$.[y] Interpreting subgroup and quotients in light of these homomorphisms emphasizes the structural concept inherent to these definitions alluded to in the introduction. In general, homomorphisms are neither injective nor surjective. Kernel and image of group homomorphisms and the first isomorphism theorem address this phenomenon.

Examples and applications



A periodic wallpaper pattern gives rise to a wallpaper group.



The fundamental group of a plane minus a point (bold) consists of loops around the missing point. This group is isomorphic to the integers.

Examples and applications of groups abound. A starting point is the group **Z** of integers with addition as group operation, introduced above. If instead of addition multiplication is considered, one obtains multiplicative groups. These groups are predecessors of important constructions in abstract algebra.

Groups are also applied in many other mathematical areas. Mathematical objects are often examined by associating groups to them and studying the properties of the corresponding groups. For example, Henri Poincaré founded what is now called algebraic topology by introducing the fundamental group.[34] By means of this connection, topological properties such as proximity and continuity translate into properties of groups.[i[›]] For example, elements of the fundamental group are represented by loops. The second image at the right shows some loops in a plane minus a point. The blue loop is considered null-homotopic (and thus irrelevant), because it can be continuously shrunk to a point. The presence of the hole prevents the orange loop from being shrunk to a point. The fundamental group of the plane with a point deleted turns out to be infinite cyclic, generated by the orange loop (or any other loop winding once around the hole). This way, the fundamental group detects the hole.

In more recent applications, the influence has also been reversed to motivate geometric constructions by a group-theoretical background.[j[›]] In a similar vein, geometric group theory employs geometric concepts, for example in the study of hyperbolic groups.[35] Further branches crucially applying groups include algebraic geometry and number theory.[36]

In addition to the above theoretical applications, many practical applications of groups exist. Cryptography relies on the combination of the abstract group theory approach together with algorithmical knowledge obtained in computational group theory, in particular when implemented for finite groups.[37] Applications of group theory are not restricted to mathematics; sciences such as physics, chemistry and computer science benefit from the concept.

Numbers

Many number systems, such as the integers and the rationals enjoy a naturally given group structure. In some cases, such as with the rationals, both addition and multiplication operations give rise to group structures. Such number systems are predecessors to more general algebraic structures known as rings and fields. Further abstract algebraic concepts such as modules, vector spaces and algebras also form groups.

Integers

The group of integers **Z** under addition, denoted (**Z**, +), has been described above. The integers, with the operation of multiplication instead of addition, (**Z**, ·) do *not* form a group. The closure, associativity and identity axioms are satisfied, but inverses do not exist: for example, $a = 2$ is an integer, but the only solution to the equation $a \cdot b = 1$ in this case is $b = 1/2$, which is a rational number, but not an integer. Hence not every element of **Z** has a (multiplicative) inverse.[k[›]]

Rationals

The desire for the existence of multiplicative inverses suggests considering fractions

$$\frac{a}{b}.$$

Fractions of integers (with b nonzero) are known as rational numbers.[l[›]] The set of all such fractions is commonly denoted **Q**. There is still a minor obstacle for (**Q**, ·), the rationals with multiplication, being a group: because the rational number 0 does not have a multiplicative inverse (i.e., there is no x such that $x \cdot 0 = 1$), (**Q**, ·) is still not a group.

However, the set of all *nonzero* rational numbers $\mathbf{Q} \setminus \{0\} = \{q \in \mathbf{Q}, q \neq 0\}$ does form an abelian group under multiplication, denoted $(\mathbf{Q} \setminus \{0\}, \cdot)$.[m[›]] Associativity and identity element axioms follow from the properties of integers. The closure requirement still holds true after removing zero, because the product of two nonzero rationals is never zero. Finally, the inverse of *a*/*b* is *b*/*a*, therefore the axiom of the inverse element is satisfied.

The rational numbers (including 0) also form a group under addition. Intertwining addition and multiplication operations yields more complicated structures called rings and—if division is possible, such as in **Q**—fields, which occupy a central position in abstract algebra. Group theoretic arguments therefore underlie parts of the theory of those entities.[n]

Nonzero integers modulo a prime

For any prime number p, modular arithmetic furnishes the multiplicative group of integers modulo p.[38] Its elements are integers not divisible by p, considered modulo p, i.e. two numbers are considered equivalent if their difference is divisible by p. For example, if $p = 5$, there are exactly four group elements 1, 2, 3, 4: multiples of 5 are excluded and 6 and −4 are both equivalent to 1 etc. The group operation is given by multiplication. Therefore, $4 \cdot 4 = 1$, because the usual product 16 is equivalent to 1, for 5 divides $16 - 1 = 15$, denoted

$$16 \equiv 1 \pmod 5.$$

The primality of p ensures that the product of two integers neither of which is divisible by p is not divisible by p either, hence the indicated set of classes is closed under multiplication.[o] The identity element is 1, as usual for a multiplicative group, and the associativity follows from the corresponding property of integers. Finally, the inverse element axiom requires that given an integer a not divisible by p, there exists an integer b such that

$$a \cdot b \equiv 1 \pmod p\text{, i.e. } p \text{ divides the difference } a \cdot b - 1.$$

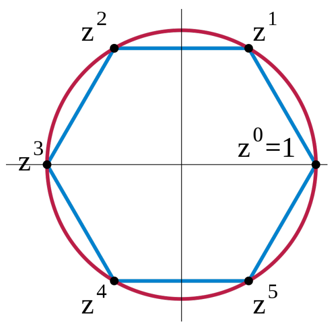
The inverse b can be found by using Bézout's identity and the fact that the greatest common divisor $\gcd(a, p)$ equals 1.[39] In the case $p = 5$ above, the inverse of 4 is 4, and the inverse of 3 is 2, as $3 \cdot 2 = 6 \equiv 1 \pmod 5$. Hence all group axioms are fulfilled. Actually, this example is similar to $(\mathbf{Q}\backslash\{0\}, \cdot)$ above, because it turns out to be the multiplicative group of nonzero elements in the finite field $\mathbf{F}_p$, denoted $\mathbf{F}_p^\times$.[40] These groups are crucial to public-key cryptography.[p]

Cyclic groups

A *cyclic group* is a group all of whose elements are powers (when the group operation is written additively, the term 'multiple' can be used) of a particular element a.[41] In multiplicative notation, the elements of the group are:

$$\ldots, a^{-3}, a^{-2}, a^{-1}, a^0 = e, a, a^2, a^3, \ldots,$$

where a^2 means $a \bullet a$, and a^{-3} stands for $a^{-1} \bullet a^{-1} \bullet a^{-1} = (a \bullet a \bullet a)^{-1}$ etc.[h] Such an element a is called a generator or a primitive element of the group.



The 6th complex roots of unity form a cyclic group. z is a primitive element, but z^2 is not, because the odd powers of z are not a power of z^2.

A typical example for this class of groups is the group of n-th complex roots of unity, given by complex numbers z satisfying $z^n = 1$ (and whose operation is multiplication).[42] Any cyclic group with n elements is isomorphic to this group. Using some field theory, the group $\mathbf{F}_p^\times$ can be shown to be cyclic: for example, if $p = 5$, 3 is a generator since $3^1 = 3$, $3^2 = 9 \equiv 4$, $3^3 \equiv 2$, and $3^4 \equiv 1$.

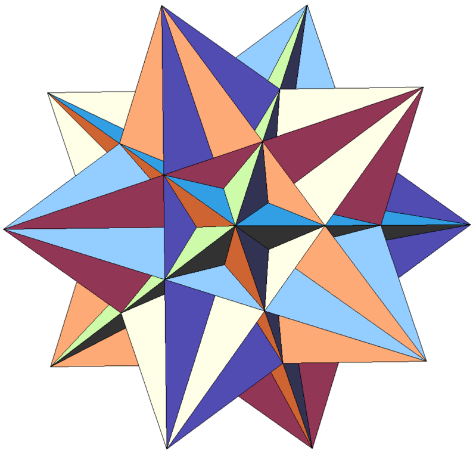
Some cyclic groups have an infinite number of elements. In these groups, for every non-zero element a, all the powers of a are distinct; despite the name "cyclic group", the powers of the elements do not cycle. An infinite cyclic group is isomorphic to $(\mathbf{Z}, +)$, the group of integers under addition introduced above.[43] As these two prototypes are both abelian, so is any cyclic group.

The study of abelian groups is quite mature, including the fundamental theorem of finitely generated abelian groups; and reflecting this state of affairs, many group-related notions, such as center and commutator, describe the extent to which a given group is not abelian.[44]

Symmetry groups

Symmetry groups are groups consisting of symmetries of given mathematical objects—be they of geometric nature, such as the introductory symmetry group of the square, or of algebraic nature, such as polynomial equations and their solutions.[45] Conceptually, group theory can be thought of as the study of symmetry.t[›] Symmetries in mathematics greatly simplify the study of geometrical or analytical objects. A group is said to act on another mathematical object X if every group element performs some operation on X compatibly to the group law. In the rightmost example below, an element of order 7 of the (2,3,7) triangle group acts on the tiling by permuting the highlighted warped triangles (and the other ones, too). By a group action, the group pattern is connected to the structure of the object being acted on.

In chemical fields, such as crystallography, space groups and point groups describe molecular symmetries and crystal symmetries. These symmetries underlie the chemical and physical behavior of these systems, and group theory enables simplification of quantum mechanical analysis of these properties.[46] For example, group theory is used to show that optical transitions between certain quantum levels cannot occur simply because of the symmetry of the states involved.

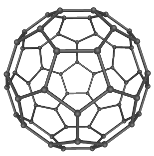
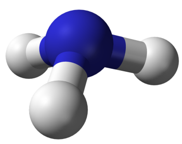
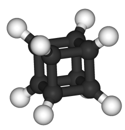
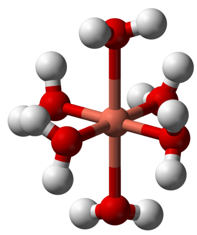
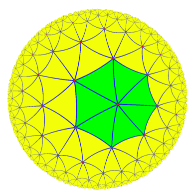


Rotations and flips form the symmetry group of a great icosahedron.

Not only are groups useful to assess the implications of symmetries in molecules, but surprisingly they also predict that molecules sometimes can change symmetry. The Jahn-Teller effect is a distortion of a molecule of high symmetry when it adopts a particular ground state of lower symmetry from a set of possible ground states that are related to each other by the symmetry operations of the molecule.[47] [48]

Likewise, group theory helps predict the changes in physical properties that occur when a material undergoes a phase transition, for example, from a cubic to a tetrahedral crystalline form. An example is ferroelectric materials, where the change from a paraelectric to a ferroelectric state occurs at the Curie temperature and is related to a change from the high-symmetry paraelectric state to the lower symmetry ferroelectic state, accompanied by a so-called soft phonon mode, a vibrational lattice mode that goes to zero frequency at the transition.[49]

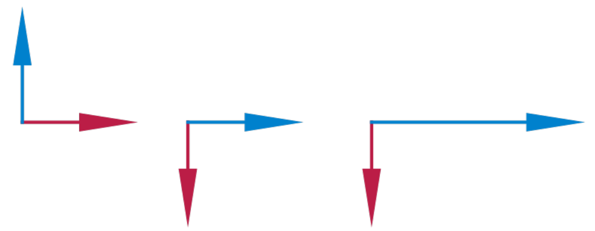
Such spontaneous symmetry breaking has found further application in elementary particle physics, where its occurrence is related to the appearance of Goldstone bosons.

				
Buckminsterfullerene displays icosahedral symmetry.	Ammonia, NH_3. Its symmetry group is of order 6, generated by a 120° rotation and a reflection.	Cubane C_8H_8 features octahedral symmetry.	Hexaaquacopper(II) complex ion, $[Cu(OH_2)_6]^{2+}$. Compared to a perfectly symmetrical shape, the molecule is vertically dilated by about 22% (Jahn-Teller effect).	The (2,3,7) triangle group, a hyperbolic group, acts on this tiling of the hyperbolic plane.

Finite symmetry groups such as the Mathieu groups are used in coding theory, which is in turn applied in error correction of transmitted data, and in CD players.[50] Another application is differential Galois theory, which characterizes functions having antiderivatives of a prescribed form, giving group-theoretic criteria for when solutions of certain differential equations are well-behaved.u[›] Geometric properties that remain stable under group actions are investigated in (geometric) invariant theory.[51]

General linear group and representation theory

Matrix groups consist of matrices together with matrix multiplication. The *general linear group* $GL(n, \mathbf{R})$ consists of all invertible n-by-n matrices with real entries.[52] Its subgroups are referred to as *matrix groups* or *linear groups*. The dihedral group example mentioned above can be viewed as a (very small) matrix group. Another important matrix group is the special orthogonal group $SO(n)$. It describes all possible rotations in n dimensions. Via Euler angles, rotation matrices are used in computer graphics.[53]



Two vectors (the left illustration) multiplied by matrices (the middle and right illustrations). The middle illustration represents a clockwise rotation by 90°, while the right-most one stretches the x-coordinate by factor 2.

Representation theory is both an application of the group concept and important for a deeper understanding of groups.[54] [55] It studies the group by its group actions on other spaces. A broad class of group representations are linear representations, i.e. the group is acting on a vector space, such as the three-dimensional Euclidean space $\mathbf{R}^3$. A representation of G on an n-dimensional real vector space is simply a group homomorphism

$$\varrho: G \to GL(n, \mathbf{R})$$

from the group to the general linear group. This way, the group operation, which may be abstractly given, translates to the multiplication of matrices making it accessible to explicit computations.w[›]

Given a group action, this gives further means to study the object being acted on.x[›] On the other hand, it also yields information about the group. Group representations are an organizing principle in the theory of finite groups, Lie groups, algebraic groups and topological groups, especially (locally) compact groups.[54] [56]

Galois groups

Galois groups have been developed to help solve polynomial equations by capturing their symmetry features.[57] [58] For example, the solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Exchanging "+" and "−" in the expression, i.e. permuting the two solutions of the equation can be viewed as a (very simple) group operation. Similar formulae are known for cubic and quartic equations, but do *not* exist in general for degree 5 and higher.[59] Abstract properties of Galois groups associated with polynomials (in particular their solvability) give a criterion for polynomials that have all their solutions expressible by radicals, i.e. solutions expressible using solely addition, multiplication, and roots similar to the formula above.[60]

The problem can be dealt with by shifting to field theory and considering the splitting field of a polynomial. Modern Galois theory generalizes the above type of Galois groups to field extensions and establishes—via the fundamental theorem of Galois theory—a precise relationship between fields and groups, underlining once again the ubiquity of groups in mathematics.

Finite groups

A group is called *finite* if it has a finite number of elements. The number of elements is called the order of the group G.[61] An important class is the *symmetric groups* S_N, the groups of permutations of N letters. For example, the symmetric group on 3 letters S_3 is the group consisting of all possible swaps of the three letters *ABC*, i.e. contains the elements *ABC*, *ACB*, ..., up to *CBA*, in total 6 (or 3 factorial) elements. This class is fundamental insofar as any finite group can be expressed as a subgroup of a symmetric group S_N for a suitable integer N (Cayley's theorem). Parallel to the group of symmetries of the square above, S_3 can also be interpreted as the group of symmetries of an equilateral triangle.

The order of an element a in a group G is the least positive integer n such that $a^n = e$, where a^n represents

$$\underbrace{a \cdots a}_{n \text{ factors}},$$

i.e. application of the operation • to n copies of a. (If • represents multiplication, then a^n corresponds to the n^{th} power of a.) In infinite groups, such an n may not exist, in which case the order of a is said to be infinity. The order of an element equals the order of the cyclic subgroup generated by this element.

More sophisticated counting techniques, for example counting cosets, yield more precise statements about finite groups: Lagrange's Theorem states that for a finite group G the order of any finite subgroup H divides the order of G. The Sylow theorems give a partial converse.

The dihedral group (discussed above) is a finite group of order 8. The order of r_1 is 4, as is the order of the subgroup R it generates (see above). The order of the reflection elements f_v etc. is 2. Both orders divide 8, as predicted by Lagrange's Theorem. The groups $\mathbf{F}_p^\times$ above have order $p - 1$.

Classification of finite simple groups

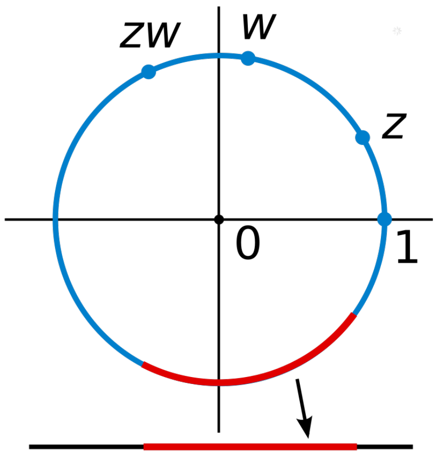
Mathematicians often strive for a complete classification (or list) of a mathematical notion. In the context of finite groups, this aim quickly leads to difficult and profound mathematics. According to Lagrange's theorem, finite groups of order p, a prime number, are necessarily cyclic (abelian) groups $\mathbf{Z}_p$. Groups of order p^2 can also be shown to be abelian, a statement which does not generalize to order p^3, as the non-abelian group D_4 of order $8 = 2^3$ above shows.[62] Computer algebra systems can be used to list small groups, but there is no classification of all finite groups.q[›] An intermediate step is the classification of finite simple groups.r[›] A nontrivial group is called *simple* if its only normal subgroups are the trivial group and the group itself.s[›] The Jordan–Hölder theorem exhibits finite simple groups as the building blocks for all finite groups.[63] Listing all finite simple groups was a major achievement in contemporary group theory. 1998 Fields Medal winner Richard Borcherds succeeded to prove the monstrous moonshine conjectures, a surprising and deep relation of the largest finite simple sporadic group—the "monster group"—with certain modular functions, a piece of classical complex analysis, and string theory, a theory supposed to unify the description of many physical phenomena.[64]

Groups with additional structure

Many groups are simultaneously groups and examples of other mathematical structures. In the language of category theory, they are group objects in a category, meaning that they are objects (that is, examples of another mathematical structure) which come with transformations (called morphisms) that mimic the group axioms. For example, every group (as defined above) is also a set, so a group is a group object in the category of sets.

Topological groups

Some topological spaces may be endowed with a group law. In order for the group law and the topology to interweave well, the group operations must be continuous functions, that is, $g \bullet h$, and g^{-1} must not vary wildly if g and h vary only little. Such groups are called *topological groups,* and they are the group objects in the category of topological spaces.[65] The most basic examples are the reals **R** under addition, (**R** \ {0}, ·), and similarly with any other topological field such as the complex numbers or p-adic numbers. All of these groups are locally compact, so they have Haar measures and can be studied via harmonic analysis. The former offer an abstract formalism of invariant integrals. Invariance means, in the case of real numbers for example:



The unit circle in the complex plane under complex multiplication is a Lie group and, therefore, a topological group. It is topological since complex multiplication and division are continuous. It is a manifold and thus a Lie group, because every small piece, such as the red arc in the figure, looks like a part of the real line (shown at the bottom).

$$\int f(x)\,dx = \int f(x+c)\,dx$$

for any constant c. Matrix groups over these fields fall under this regime, as do adele rings and adelic algebraic groups, which are basic to number theory.[66] Galois groups of infinite field extensions such as the absolute Galois group can also be equipped with a topology, the so-called Krull topology, which in turn is central to generalize the above sketched connection of fields and groups to infinite field extensions.[67] An advanced generalization of this idea, adapted to the needs of algebraic geometry, is the étale fundamental group.[68]

Lie groups

Lie groups (in honor of Sophus Lie) are groups which also have a manifold structure, i.e. they are spaces looking locally like some Euclidean space of the appropriate dimension.[69] Again, the additional structure, here the manifold structure, has to be compatible, i.e. the maps corresponding to multiplication and the inverse have to be smooth.

A standard example is the general linear group introduced above: it is an open subset of the space of all n-by-n matrices, because it is given by the inequality

$$\det(A) \neq 0,$$

where A denotes an n-by-n matrix.[70]

Lie groups are of fundamental importance in physics: Noether's theorem links continuous symmetries to conserved quantities.[71] Rotation, as well as translations in space and time are basic symmetries of the laws of mechanics.

They can, for instance, be used to construct simple models—imposing, say, axial symmetry on a situation will typically lead to significant simplification in the equations one needs to solve to provide a physical description.v[›] Another example are the Lorentz transformations, which relate measurements of time and velocity of two observers in motion relative to each other. They can be deduced in a purely group-theoretical way, by expressing the transformations as a rotational symmetry of Minkowski space. The latter serves—in the absence of significant gravitation—as a model of space time in special relativity.[72] The full symmetry group of Minkowski space, i.e. including translations, is known as the Poincaré group. By the above, it plays a pivotal role in special relativity and, by implication, for quantum field theories.[73] Symmetries that vary with location are central to the modern description of physical interactions with the help of gauge theory.[74]

Generalizations

Group-like structures				
	Totality	**Associativity**	**Identity**	**Inverses**
Group	Yes	Yes	Yes	Yes
Monoid	Yes	Yes	Yes	No
Semigroup	Yes	Yes	No	No
Loop	Yes	No	Yes	Yes
Quasigroup	Yes	No	No	Yes
Magma	Yes	No	No	No
Groupoid	No	Yes	Yes	Yes
Category	No	Yes	Yes	No

In abstract algebra, more general structures are defined by relaxing some of the axioms defining a group.[26] [75] [76] For example, if the requirement that every element has an inverse is eliminated, the resulting algebraic structure is called a monoid. The natural numbers **N** (including 0) under addition form a monoid, as do the nonzero integers under multiplication ($\mathbf{Z} \setminus \{0\}, \cdot$), see above. There is a general method to formally add inverses to elements to any (abelian) monoid, much the same way as ($\mathbf{Q} \setminus \{0\}, \cdot$) is derived from ($\mathbf{Z} \setminus \{0\}, \cdot$), known as the Grothendieck group. Groupoids are similar to groups except that the composition $a \bullet b$ need not be defined for all a and b. They arise in the study of more complicated forms of symmetry, often in topological and analytical structures, such as the fundamental groupoid. Finally, it is possible to generalize any of these concepts by replacing the binary operation with an arbitrary n-ary one (i.e. an operation taking n arguments). With the proper generalization of the group axioms this gives rise to an n-ary group.[77] The table gives a list of several structures generalizing groups.

See also

- Group ring
- Group algebra
- Euclidean group
- Free group
- Finitely presented group
- Fundamental group
- Grothendieck group
- Symmetry in physics

Notes

^ a: Mathematical Reviews lists 3,224 research papers on group theory and its generalizations written in 2005.

^ b: The closure axiom is already implied by the condition that • be a binary operation. Some authors therefore omit this axiom. Lang 2002

^ c: See, for example, the books of Lang (2002, 2005) and Herstein (1996, 1975).

^ d: However, a group is not determined by its lattice of subgroups. See Suzuki 1951.

^ e: The fact that the group operation extends this canonically is an instance of a universal property.

^ f: For example, if *G* is finite, then the size of any subgroup and any quotient group divides the size of *G*, according to Lagrange's theorem.

^ g: The word homomorphism derives from Greek ὁμός—the same and μορφή—structure.

^ h: The additive notation for elements of a cyclic group would be $t \cdot a$, t in **Z**.

^ i: See the Seifert–van Kampen theorem for an example.

^ j: An example is group cohomology of a group which equals the singular homology of its classifying space.

^ k: Elements which do have multiplicative inverses are called units, see Lang 2002, §II.1, p. 84.

^ l: The transition from the integers to the rationals by adding fractions is generalized by the quotient field.

^ m: The same is true for any field *F* instead of **Q**. See Lang 2005, §III.1, p. 86.

^ n: For example, a finite subgroup of the multiplicative group of a field is necessarily cyclic. See Lang 2002, Theorem IV.1.9. The notions of torsion of a module and simple algebras are other instances of this principle.

^ o: The stated property is a possible definition of prime numbers. See prime element.

^ p: For example, the Diffie-Hellman protocol uses the discrete logarithm.

^ q: The groups of order at most 2000 are known. Up to isomorphism, there are about 49 billion. See Besche, Eick & O'Brien 2001.

^ r: The gap between the classification of simple groups and the one of all groups lies in the extension problem, a problem too hard to be solved in general. See Aschbacher 2004, p. 737.

^ s: Equivalently, a nontrivial group is simple if its only quotient groups are the trivial group and the group itself. See Michler 2006, Carter 1989.

^ t: More rigorously, every group is the symmetry group of some graph, see Frucht 1939.

^ u: More precisely, the monodromy action on the vector space of solutions of the differential equations is considered. See Kuga 1993, pp. 105–113.

^ v: See Schwarzschild metric for an example where symmetry greatly reduces the complexity of physical systems.

^ w: This was crucial to the classification of finite simple groups, for example. See Aschbacher 2004.

^ x: See, for example, Schur's Lemma for the impact of a group action on simple modules. A more involved example is the action of an absolute Galois group on étale cohomology.

^ y: Injective and surjective maps correspond to mono- and epimorphisms, respectively. They are interchanged when passing to the dual category.

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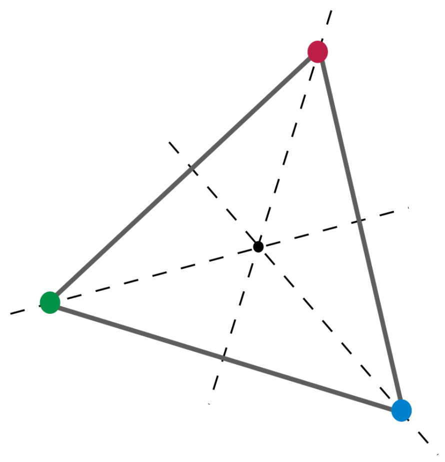
Group action



Given an equilateral triangle, the counterclockwise rotation by 120° around the center of the triangle "acts" on the set of vertices of the triangle by mapping every vertex to another one.

In algebra and geometry, a **group action** is a way of describing symmetries of objects using groups. The essential elements of the object are described by a set and the symmetries of the object are described by the symmetry group of this set, which consists of bijective transformations of the set. In this case, the group is also called a **permutation group** (especially if the set is finite or not a vector space) or **transformation group** (especially if the set is a vector space and the group acts like linear transformations of the set).

A group action is a flexible generalization of the notion of a symmetry group in which every element of the group "acts" like a bijective transformation (or "symmetry") of some set, without being identified with that transformation. This allows for a more comprehensive description of the symmetries of an object, such as a polyhedron, by allowing the same group to act on several different sets, such as the set of vertices, the set of edges and the set of faces of the polyhedron.

If G is a group and X is a set then a group action may be defined as a group homomorphism from G to the symmetric group of X. The action assigns a permutation of X to each element of the group in such a way that

- the permutation of X assigned to the identity element of G is the identity transformation of X;
- the permutation of X assigned to a product gh of two elements of the group is the composite of the permutations assigned to g and h.

Since each element of G is represented as a permutation, a group action is also known as a **permutation representation**.

The abstraction provided by group actions is a powerful one, because it allows geometrical ideas to be applied to more abstract objects. Many objects in mathematics have natural group actions defined on them. In particular, groups can act on other groups, or even on themselves. Despite this generality, the theory of group actions contains wide-reaching theorems, such as the orbit stabilizer theorem, which can be used to prove deep results in several fields.

Definition

If G is a group and X is a set, then a (**left**) **group action** of G on X is a binary function

$$G \times X \to X$$

denoted

$$(g, x) \mapsto g \cdot x$$

which satisfies the following two axioms:

1. $(gh)\cdot x = g\cdot(h\cdot x)$ for all g, h in G and x in X;
2. $e\cdot x = x$ for every x in X (where e denotes the identity element of G).

The set X is called a (**left**) ***G*-set**. The group G is said to act on X (on the left).

From these two axioms, it follows that for every g in G, the function which maps x in X to $g\cdot x$ is a bijective map from X to X (its inverse being the function which maps x to $g^{-1}\cdot x$). Therefore, one may alternatively define a group action of G on X as a group homomorphism from G into the symmetric group Sym(X) of all bijections from X to X.

In complete analogy, one can define a **right group action** of G on X as a function $X \times G \to X$ by the two axioms:

1. $x \cdot (gh) = (x \cdot g) \cdot h$;
2. $x \cdot e = x$.

The difference between left and right actions is in the order in which a product like gh acts on x. For a left action h acts first and is followed by g, while for a right action g acts first and is followed by h. From a right action a left action can be constructed by composing with the inverse operation on the group. If r is a right action, then

$$l : G \times X \to X : (g, x) \mapsto r(x, g^{-1})$$

is a left action, since

$$\begin{aligned} l(gh, x) &= r(x, (gh)^{-1}) = x \cdot (h^{-1} g^{-1}) \\ &= (x \cdot h^{-1}) \cdot g^{-1} = l(h, x) \cdot g^{-1} = l(g, l(h, x)) \end{aligned}$$

and

$$l(e, x) = r(x, e^{-1}) = x \cdot e = x.$$

Similarly, any left action can be converted into a right action. Therefore in the sequel we consider only left group actions, since right actions add nothing new.

Examples

- The **trivial** action for any group G is defined by $g \cdot x = x$ for all g in G and all x in X; that is, the whole group G induces the identity permutation on X.
- Every group G acts on G in two natural but essentially different ways: $g \cdot x = gx$ for all x in G, or $g \cdot x = gxg^{-1}$ for all x in G. The latter action is often called the conjugation action, and an exponential notation is commonly used for the right-action variant: $x^g = g^{-1}xg$; it satisfies $(x^g)^h = x^{gh}$.
- The symmetric group S_n and its subgroups act on the set { 1, ... , n } by permuting its elements
- The symmetry group of a polyhedron acts on the set of vertices of that polyhedron. It also acts on the set of faces of the polyhedron.
- The symmetry group of any geometrical object acts on the set of points of that object
- The automorphism group of a vector space (or graph, or group, or ring...) acts on the vector space (or set of vertices of the graph, or group, or ring...).
- The general linear group GL(n,**R**), special linear group SL(n,**R**), orthogonal group O(n,**R**), and special orthogonal group SO(n,**R**) are Lie groups which act on $\mathbf{R}^n$.
- The Galois group of a field extension E/F acts on the bigger field E. So does every subgroup of the Galois group.
- The additive group of the real numbers (**R**, +) acts on the phase space of "well-behaved" systems in classical mechanics (and in more general dynamical systems): if t is in **R** and x is in the phase space, then x describes a state of the system, and $t \cdot x$ is defined to be the state of the system t seconds later if t is positive or $-t$ seconds ago if t is negative.
- The additive group of the real numbers (**R**, +) acts on the set of real functions of a real variable with $(g \cdot f)(x)$ equal to e.g. $f(x + g)$, $f(x) + g$, $f(xe^g)$, $f(x)e^g$, $f(x + g)e^g$, or $f(xe^g) + g$, but not $f(xe^g + g)$
- The quaternions with modulus 1, as a multiplicative group, act on $\mathbf{R}^3$: for any such quaternion $z = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\,\hat{\mathbf{v}}$, the mapping $f(\mathbf{x}) = z\,\mathbf{x}\,z^*$ is a counterclockwise rotation through an angle α about an axis **v**; $-z$ is the same rotation; see quaternions and spatial rotation.
- The isometries of the plane act on the set of 2D images and patterns, such as a wallpaper pattern. The definition can be made more precise by specifying what is meant by image or pattern, e.g. a function of position with values in a set of colors.
- More generally, a group of bijections g: V → V acts on the set of functions x: $V \to W$ by $(gx)(v) = x(g^{-1}(v))$ (or a restricted set of such functions that is closed under the group action). Thus a group of bijections of space induces

a group action on "objects" in it.

Types of actions

The action of G on X is called

- **transitive** if X is non-empty and for any two x, y in X there exists a g in G such that $g{\cdot}x = y$.
  - **sharply transitive** if that g is unique; it is equivalent to regularity defined below.
- ***n*-transitive** if X has at least n elements and for any pairwise distinct $x_1, ..., x_n$ and pairwise distinct $y_1, ..., y_n$ there is a g in G such that $g.x_k = y_k$ for $1 \le k \le n$. A 2-transitive action is also called **doubly transitive**, a 3-transitive action is also called **triply transitive**, and so on. Such actions define 2-transitive groups, 3-transitive groups, and multiply transitive groups.
  - **sharply *n*-transitive** if there is exactly one such g. See also sharply triply transitive groups.
- **faithful** (or **effective**) if for any two distinct g, h in G there exists an x in X such that $g{\cdot}x \ne h{\cdot}x$; or equivalently, if for any $g \ne e$ in G there exists an x in X such that $g{\cdot}x \ne x$. Intuitively, different elements of G induce different permutations of X.
- **free** (or **semiregular**) if for any two distinct g, h in G and all x in X we have $g{\cdot}x \ne h{\cdot}x$; or equivalently, if $g{\cdot}x = x$ for some x then $g = e$.
- **regular** (or **simply transitive**) if it is both transitive and free; this is equivalent to saying that for any two x, y in X there exists precisely one g in G such that $g{\cdot}x = y$. In this case, X is known as a principal homogeneous space for G or as a G-torsor.
- **locally free** if G is a topological group, and there is a neighbourhood U of e in G such that the restriction of the action to U is free; that is, if $g{\cdot}x = x$ for some x and some g in U then $g = e$.
- **irreducible** if X is a nonzero module over a ring R, the action of G is R-linear, and there is no nonzero proper invariant submodule.

Every free action on a non-empty set is faithful. A group G acts faithfully on X if and only if the homomorphism $G \to \mathrm{Sym}(X)$ has a trivial kernel. Thus, for a faithful action, G is isomorphic to a permutation group on X; specifically, G is isomorphic to its image in $\mathrm{Sym}(X)$.

The action of any group G on itself by left multiplication is regular, and thus faithful as well. Every group can, therefore, be embedded in the symmetric group on its own elements, $\mathrm{Sym}(G)$ — a result known as Cayley's theorem.

If G does not act faithfully on X, one can easily modify the group to obtain a faithful action. If we define $N = \{g \text{ in } G : g{\cdot}x = x \text{ for all } x \text{ in } X\}$, then N is a normal subgroup of G; indeed, it is the kernel of the homomorphism $G \to \mathrm{Sym}(X)$. The factor group G/N acts faithfully on X by setting $(gN){\cdot}x = g{\cdot}x$. The original action of G on X is faithful if and only if $N = \{e\}$.

Orbits and stabilizers

Consider a group G acting on a set X. The **orbit** of a point x in X is the set of elements of X to which x can be moved by the elements of G. The orbit of x is denoted by Gx:

$$Gx = \{g \cdot x \mid g \in G\}.$$

The defining properties of a group guarantee that the set of orbits of (points x in) X under the action of G form a partition of X. The associated equivalence relation is defined by saying $x \sim y$ if and only if there exists a g in G with $g{\cdot}x = y$. The orbits are then the equivalence classes under this relation; two elements x and y are equivalent if and only if their orbits are the same, i.e. $Gx = Gy$.

The set of all orbits of X under the action of G is written as X/G (or, less frequently: $G\backslash X$), and is called the ***quotient*** of the action; in geometric situations it may be called the ***orbit space***.

If Y is a subset of X, we write GY for the set $\{ g \cdot y : y \in Y \text{ and } g \in G \}$. We call the subset Y *invariant under G* if $GY = Y$ (which is equivalent to $GY \subseteq Y$). In that case, G also operates on Y. The subset Y is called *fixed under G* if $g \cdot y = y$ for all g in G and all y in Y. Every subset that's fixed under G is also invariant under G, but not vice versa.

Every orbit is an invariant subset of X on which G acts transitively. The action of G on X is **transitive** if and only if all elements are equivalent, meaning that there is only one orbit.

For every x in X, we define the **stabilizer subgroup** of x (also called the **isotropy group** or **little group**) as the set of all elements in G that fix x:

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

This is a subgroup of G, though typically not a normal one. The action of G on X is free if and only if all stabilizers are trivial. The kernel N of the homomorphism $G \to \mathrm{Sym}(X)$ is given by the intersection of the stabilizers G_x for all x in X.

Orbits and stabilizers are closely related. For a fixed x in X, consider the map from G to X given by $g \mapsto g \cdot x$. The image of this map is the orbit of x and the coimage is the set of all left cosets of G_x. The standard quotient theorem of set theory then gives a natural bijection between G / G_x and Gx. Specifically, the bijection is given by $hG_x \mapsto h \cdot x$. This result is known as the **orbit-stabilizer theorem**.

If G and X are finite then the orbit-stabilizer theorem, together with Lagrange's theorem, gives

$$|Gx| = [G \, : \, G_x] = |G|/|G_x|.$$

This result is especially useful since it can be employed for counting arguments.

Note that if two elements x and y belong to the same orbit, then their stabilizer subgroups, G_x and G_y, are isomorphic (or conjugate). More precisely: if $y = g \cdot x$, then $G_y = gG_x g^{-1}$. Points with conjugate stabilizer subgroups are said to have the same **orbit-type**.

A result closely related to the orbit-stabilizer theorem is Burnside's lemma:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

where X^g is the set of points fixed by g. This result is mainly of use when G and X are finite, when it can be interpreted as follows: the number of orbits is equal to the average number of points fixed per group element.

The set of formal differences of finite G-sets forms a ring called the Burnside ring, where addition corresponds to disjoint union, and multiplication to Cartesian product.

Group actions and groupoids

The notion of group action can be put in a broader context by using the associated `action groupoid' $G' = G \ltimes X$ associated to the group action, thus allowing techniques from groupoid theory such as presentations and fibrations. Further the stabilisers of the action are the vertex groups, and the orbits of the action are the components, of the action groupoid. For more details, see the book `Topology and groupoids' referenced below.

This action groupoid comes with a morphism $p : G' \to G$ which is a `covering morphism of groupoids'. This allows a relation between such morphisms and covering maps in topology.

Morphisms and isomorphisms between *G*-sets

If *X* and *Y* are two *G*-sets, we define a *morphism* from *X* to *Y* to be a function $f : X \to Y$ such that $f(g\cdot x) = g\cdot f(x)$ for all *g* in *G* and all *x* in *X*. Morphisms of *G*-sets are also called *equivariant maps* or *G-maps*.

If such a function *f* is bijective, then its inverse is also a morphism, and we call *f* an *isomorphism* and the two *G*-sets *X* and *Y* are called *isomorphic*; for all practical purposes, they are indistinguishable in this case.

Some example isomorphisms:

- Every regular *G* action is isomorphic to the action of *G* on *G* given by left multiplication.
- Every free *G* action is isomorphic to $G\times S$, where *S* is some set and *G* acts by left multiplication on the first coordinate.
- Every transitive *G* action is isomorphic to left multiplication by *G* on the set of left cosets of some subgroup *H* of *G*.

With this notion of morphism, the collection of all *G*-sets forms a category; this category is a Grothendieck topos (in fact, assuming a classical metalogic, this topos will even be Boolean).

Continuous group actions

One often considers **continuous group actions**: the group *G* is a topological group, *X* is a topological space, and the map $G \times X \to X$ is continuous with respect to the product topology of $G \times X$. The space *X* is also called a *G-space* in this case. This is indeed a generalization, since every group can be considered a topological group by using the discrete topology. All the concepts introduced above still work in this context, however we define morphisms between *G*-spaces to be *continuous* maps compatible with the action of *G*. The quotient *X/G* inherits the quotient topology from *X*, and is called the **quotient space** of the action. The above statements about isomorphisms for regular, free and transitive actions are no longer valid for continuous group actions.

If *G* is a discrete group acting on a topological space *X*, the action is properly discontinuous if for any point *x* in *X* there is an open neighborhood *U* of *x* in *X*, such that the set of all $g \in G$ for which $g(U) \cap U \neq \emptyset$ consists of the identity only. If *X* is a regular covering space of another topological space *Y*, then the action of the deck transformation group on *X* is properly discontinuous as well as being free. Every free, properly discontinuous action of a group *G* on a path-connected topological space *X* arises in this manner: the quotient map $X \mapsto X/G$ is a regular covering map, and the deck transformation group is the given action of *G* on *X*. Furthermore, if *X* is simply connected, the fundamental group of X/G will be isomorphic to G . These results have been generalised in the book *Topology and Groupoids* referenced below to obtain the fundamental groupoid of the orbit space of a discontinuous action of discrete group on a Hausdorff space, as, under reasonable local conditions, the orbit groupoid of the fundamental groupoid of the space. This allows calculations such as the fundamental group of a symmetric square.
An action of a group *G* on a locally compact space *X* is **cocompact** if there exists a compact subset *A* of *X* such that $GA = X$. For a properly discontinuous action, cocompactness is equivalent to compactness of the quotient space *X/G*.

The action of *G* on *X* is said to be **proper** if the mapping $G\times X \to X\times X$ that sends $(g,x)\mapsto(gx,x)$ is a proper map.

Strongly continuous group action and smooth points

If $\alpha : G \times X \to X$ is an action of a topological group G on another topological space X , one says that it is **strongly continuous** if for all $x \in X$, the map $g \mapsto \alpha_g x$ is continuous with respect to the respective topologies. Such an action induce an action on the space of continuous function on X by $(\alpha_g f)(x) = f(\alpha_g^{-1} x)$.

The subspace of **smooth points** for the action α is the subspace of X of points x such that $g \mapsto \alpha_g(x)$ is smooth, i.e. it is continuous and all derivatives are continuous.

Generalizations

One can also consider actions of monoids on sets, by using the same two axioms as above. This does not define bijective maps and equivalence relations however.

Instead of actions on sets, one can define actions of groups and monoids on objects of an arbitrary category: start with an object *X* of some category, and then define an action on *X* as a monoid homomorphism into the monoid of endomorphisms of *X*. If *X* has an underlying set, then all definitions and facts stated above can be carried over. For example, if we take the category of vector spaces, we obtain group representations in this fashion.

One can view a group *G* as a category with a single object in which every morphism is invertible. A group action is then nothing but a functor from *G* to the category of sets, and a group representation is a functor from *G* to the category of vector spaces. A morphism between G-sets is then a natural transformation between the group action functors. In analogy, an action of a groupoid is a functor from the groupoid to the category of sets or to some other category.

Without using the language of categories, one can extend the notion of a group action on a set *X* by studying as well its induced action on the power set of *X*. This is useful, for instance, in studying the action of the large Mathieu group on a 24-set and in studying symmetry in certain models of finite geometries.

See also

- Group with operators
- Monoid action
- Gain graph

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Field (mathematics)

In abstract algebra, a **field** is an algebraic structure with notions of addition, subtraction, multiplication, and division, satisfying certain axioms. The most commonly used fields are the field of real numbers, the field of complex numbers, and the field of rational numbers, but there are also finite fields, fields of functions, various algebraic number fields, *p*-adic fields, and so forth.

Any field may be used as the scalars for a vector space, which is the standard general context for linear algebra. The theory of field extensions (including Galois theory) involves the roots of polynomials with coefficients in a field; among other results, this theory leads to impossibility proofs for the classical problems of angle trisection and squaring the circle with a compass and straightedge, as well as a proof of the Abel–Ruffini theorem on the insolubility of quintic equations. In modern mathematics, the theory of fields (or **field theory**) plays an essential role in number theory and algebraic geometry.

As an algebraic structure, every field is a ring, but not every ring is a field. The most important difference is that fields allow for division (though not division by zero), while a ring need not possess multiplicative inverses. Also, the multiplication operation in a field is required to be commutative. A ring in which division is possible but commutativity is not assumed (such as the quaternions) is called a *division ring* or *skew field*. (Historically, division rings were sometimes referred to as fields, while fields were called “commutative fields”.)

As a ring, a field may be classified as a specific type of integral domain, and can be characterized by the following (not necessarily exhaustive) chain of class inclusions:

- **Commutative rings** ⊃ **integral domains** ⊃ **integrally closed domains** ⊃ **unique factorization domains** ⊃ **principal ideal domains** ⊃ **Euclidean domains** ⊃ **fields** ⊃ **finite fields**.

Definition and illustration

An example of a field is the set **Q** of rational numbers. In **Q**, there are four essential operations: addition together with subtraction, and multiplication with division. Intuitively, a field is a set of numbers which has four such operations. In order to qualify as a field, these operations have to satisfy certain axioms.

Intuitively, a field is a set F that is a commutative group with respect to two compatible operations, addition and multiplication, with "compatible" being formalized by *distributivity,* and the caveat that the additive identity (0) has no multiplicative inverse (one cannot divide by 0).

The most common way to formalize this is by defining a *field* as a set together with two operations, usually called *addition* and *multiplication*, and denoted by + and ·, respectively, such that the following axioms hold; *subtraction* and *division* are defined implicitly in terms of the inverse operations of addition and multiplication:[1]

Closure of F under addition and multiplication

For all a, b in F, both $a + b$ and $a \cdot b$ are in F (or more formally, + and · are binary operations on F).

Associativity of addition and multiplication

For all a, b, and c in F, the following equalities hold: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Commutativity of addition and multiplication

For all a and b in F, the following equalities hold: $a + b = b + a$ and $a \cdot b = b \cdot a$.

Additive and multiplicative *identity*

There exists an element of F, called the *additive identity* element and denoted by 0, such that for all a in F, $a + 0 = a$. Likewise, there is an element, called the *multiplicative identity* element and denoted by 1, such that for all a in F, $a \cdot 1 = a$. For technical reasons, the additive identity and the multiplicative identity are required to be distinct.

Additive and multiplicative *inverses*

For every a in F, there exists an element $-a$ in F, such that $a + (-a) = 0$. Similarly, for any a in F other than 0, there exists an element a^{-1} in F, such that $a \cdot a^{-1} = 1$. (The elements $a + (-b)$ and $a \cdot b^{-1}$ are also denoted $a - b$ and a/b, respectively.) In other words, *subtraction* and *division* operations exist.

Distributivity of multiplication over addition

For all a, b and c in F, the following equality holds: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Note that all but the last axiom are exactly the axioms for a commutative group, while the last axiom is a compatibility condition between the two operations.

First example: rational numbers

A simple example of a field is the field of rational numbers, consisting of the fractions a/b, where a and b are integers, and $b \neq 0$. The additive inverse of such a fraction is simply $-a/b$, and the multiplicative inverse—provided that $a \neq 0$, as well—is b/a. To see the latter, note that

$$\frac{b}{a} \cdot \frac{a}{b} = \frac{ba}{ab} = 1.$$

The abstractly required field axioms reduce to standard properties of rational numbers, such as the law of distributivity

$$\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{cf + ed}{df} = \frac{a(cf + ed)}{bdf} = \frac{acf}{bdf} + \frac{aed}{bdf} = \frac{ac}{bd} + \frac{ae}{bf} = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f},$$

or the law of commutativity and law of associativity.

Second example: a field with four elements

+	**O**	**I**	**A**	**B**
O	O	I	A	B
I	I	O	B	A
A	A	B	O	I
B	B	A	I	O

·	**O**	**I**	**A**	**B**
O	O	O	O	O
I	O	I	A	B
A	O	A	B	I
B	O	B	I	A

In addition to familiar number systems such as the rationals, there are other, less immediate examples of fields. The following example is a field consisting of four elements called O, I, A and B. The notation is chosen such that O plays the role of the additive identity element (denoted 0 in the axioms), and I is the multiplicative identity (denoted 1 above). Checking that all field axioms are indeed satisfied is easy, if tedious. For example:

A · (B + A) = A · I = A, which equals A · B + A · A = I + B = A, as required by the distributivity.

The above field is called a finite field with four elements, denoted $\mathbf{F}_4$. Field theory is concerned with understanding the reasons for the existence of this field, defined in a fairly ad-hoc manner, and with describing its inner structure. For example, from a glance at the multiplication table, it can be seen that any non-zero element, i.e., I, A, and B, is a power of A. Indeed $A = A^1$, $B = A^2 = A \cdot A$, and finally $I = A^3 = A \cdot A \cdot A$. This is no coincidence, but one of the starting points of a deeper understanding of (finite) fields.

Alternative axiomatizations

As with other algebraic structures, there exist alternative axiomatizations. Because of the relations between the operations, one can alternatively axiomatize a field by explicitly assuming that are four binary operations (add, subtract, multiply, divide) with axioms relating these, or in terms of two binary operations (add, multiply) and two unary operations (additive inverse, multiplicative inverse), or other variants.

The usual axiomatization in terms of the two operations of addition and multiplication is brief and allows the other operations to be defined in terms of these basic ones, but in other contexts, such as topology and category theory, it is important to include all operations as explicitly given, rather than implicitly defined (compare topological group). This is because without further assumptions, the implicitly defined inverses may not be continuous (in topology), or may not be able to be defined (in category theory): defining an inverse requires that one be working with a set, not a more general object.

Related algebraic structures

Ring and field axioms					
	Abelian group	**Ring**	**Commutative ring**	**Skew field or Division ring**	**Field**
Abelian (additive) group structure	Yes	Yes	Yes	Yes	Yes
Multiplicative structure and distributivity	–	Yes	Yes	Yes	Yes
Commutativity of multiplication	–	No	Yes	No	Yes
Multiplicative inverses	–	No	No	Yes	Yes

The axioms imposed above resemble the ones familiar from other algebraic structures. For example, the existence of the binary operation "·", together with its commutativity, associativity, (multiplicative) identity element and inverses are precisely the axioms for an abelian group. In other words, for any field, the subset of nonzero elements $F \setminus \{0\}$, also often denoted $F^{\times}$, is an abelian group ($F^{\times}$, ·) usually called multiplicative group of the field. Likewise (F, +) is an abelian group. The structure of a field is hence the same as specifying such two group structures (on the same set), obeying the distributivity.

Important other algebraic structures such as rings arise when requiring only part of the above axioms. For example, if the requirement of commutativity of the multiplication operation · is dropped, one gets structures usually called division rings or *skew fields*.

Remarks

By elementary group theory, applied to the abelian groups ($F^{\times}$, ·), and (F, +), the additive inverse $-a$ and the multiplicative inverse a^{-1} are uniquely determined by a.

Similar direct consequences from the field axioms include

$$-(a \cdot b) = (-a) \cdot \mathrm{b} = a \cdot (-b), \text{ in particular } -a = (-1) \cdot a$$

as well as

$$a \cdot 0 = 0.$$

Both can be shown by replacing b or c with 0 in the distributive property

History

The concept of *field* was used implicitly by Niels Henrik Abel and Évariste Galois in their work on the solvability of polynomial equations with rational coefficients of degree 5 or higher.

In 1871, Richard Dedekind introduced, for a set of real or complex numbers which is closed under the four arithmetic operations, the German word *Körper*, which means "body" or "corpus" (to suggest an organically closed entity), hence the common use of the letter K to denote a field. He also defined rings (then called *order* or *order-modul*), but the term *"a ring"* (*Zahlring*) was invented by Hilbert. [2] In 1893, Eliakim Hastings Moore called the concept "field" in English.[3]

In 1881, Leopold Kronecker defined what he called a "domain of rationality", which is indeed a field of polynomials in modern terms. In 1893, Heinrich M. Weber gave the first clear definition of an abstract field.[4] In 1910 Ernst Steinitz published the very influential paper *Algebraische Theorie der Körper* (English: *Algebraic Theory of Fields*).[5] In this paper he axiomatically studies the properties of fields and defines many important field theoretic concepts like prime field, perfect field and the transcendence degree of a field extension.

Emil Artin developed the relationship between groups and fields in great detail during 1928-1942.

Examples

Rationals and algebraic numbers

The field of rational numbers **Q** has been introduced above. A related class of fields very important in number theory are algebraic number fields. We will first give an example, namely the field $\mathbf{Q}[\zeta_3]$ consisting of expressions

$$a + b \cdot \zeta + c \cdot \zeta^2, a, b, c \in \mathbf{Q}$$

where ζ is a third root of unity, i.e., a complex number satisfying $\zeta^3 = 1$, $\zeta \neq 1$, can be used to prove a special case of Fermat's last theorem, which asserts the non-existence of rational nonzero solutions to the equation

$$x^3 + y^3 = z^3.$$

In the language of field extensions detailed below, $\mathbf{Q}[\zeta_3]$ is a field extension of degree 3. Algebraic number fields are by definition finite field extensions of **Q**, that is, fields containing **Q** having finite dimension as a **Q**-vector space.

Reals, complex numbers, and p-adic numbers

Take the real numbers **R**, under the usual operations of addition and multiplication. When the real numbers are given the usual ordering, they form a *complete ordered field*; it is this structure which provides the foundation for most formal treatments of calculus.

The complex numbers **C** consist of expressions

$a + b\mathrm{i}$

where i is the imaginary unit, i.e., a (non-real) number satisfying $\mathrm{i}^2 = -1$. Addition and multiplication of real numbers are defined in such a way that all field axioms hold for **C**. For example, the distributive law enforces

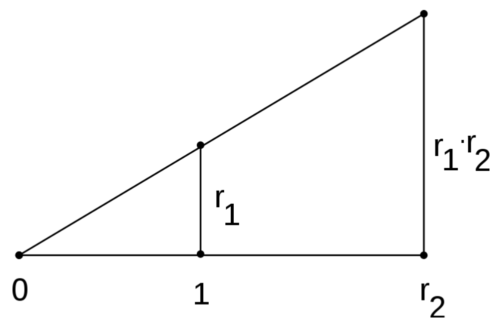
$(a + b\mathrm{i})\cdot(c + d\mathrm{i}) = ac + bc\mathrm{i} + ad\mathrm{i} + bd\mathrm{i}^2$, which equals $ac-bd + (bc + ad)\mathrm{i}$.

The real numbers can be constructed by completing the rational numbers, i.e., filling the "gaps": for example $\sqrt{2}$ is such a gap. By a formally very similar procedure, another important class of fields, the field of p-adic numbers $\mathbf{Q}_p$ is built. It is used in number theory and p-adic analysis.

Hyperreal numbers and superreal numbers extend the real numbers with the addition of infinitesimal and infinite numbers.

Constructible numbers

In antiquity, several geometric problems concerned the (in)feasibility of constructing certain numbers with compass and straightedge. For example it was unknown to the Greeks that it is in general impossible to trisect a given angle. Using the field notion and field theory allows these these problems to be settled. To do so, the field of constructible numbers is considered. It contains, on the plane, the points 0 and 1, and all complex numbers that can be constructed from these two by a finite number of construction steps using only compass and straightedge. This set, endowed with the usual addition and multiplication of complex numbers does form a field. For example, multiplying two (real) numbers r_1 and r_2 that have already been constructed can be done using construction at the right, based on the intercept theorem. This way, the obtained field F contains all rational numbers, but is bigger than **Q**, because for any $f \in F$, the square root of f is also a constructible number.



Given 0, 1, r_1 and r_2, the construction yields $r_1 \cdot r_2$

Finite fields

Finite fields (also called *Galois fields*) are fields with finitely many elements. The above introductory example $\mathbf{F}_4$ is a field with four elements. Highlighted in the multiplication and addition tables above is the field $\mathbf{F}_2$ consisting of two elements O and I. This is the smallest field, because by definition a field has at least two distinct elements $1 \neq 0$. Interpreting the addition and multiplication in this latter field as XOR and AND operations, this field finds applications in computer science, especially in cryptography and coding theory.

In a finite field there is necessarily an integer n such that $1 + 1 + ... + 1$ (n repeated terms) equals 0. It can be shown that the smallest such n must be a prime number, called the *characteristic* of the field. If a (necessarily infinite) field has the property that $1 + 1 + ... + 1$ is never zero, for any number of summands, such as in **Q**, for example, the characteristic is said to be zero.

A basic class of finite fields are the fields $\mathbf{F}_p$ with p elements (p a prime number):

$\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z} = \{0, 1, ..., p - 1\}$, where the operations are defined by performing the operation in the set of integers **Z**, dividing by p and taking the remainder; see modular arithmetic. A field **K** of characteristic p necessarily contains $\mathbf{F}_p$, and therefore may be viewed as a vector space over $\mathbf{F}_p$, of finite dimension if **K** is finite. Thus a finite field **K** has prime power order, i.e., **K** has $q = p^n$ elements (where $n > 0$ is the number of elements in a basis of **K** over $\mathbf{F}_p$). By developing more field theory, in particular the notion of the splitting field of a polynomial f over a field **K**, which is the smallest field containing **K** and all roots of f, one can show that two finite fields with the same number of elements are isomorphic, i.e., there is a one-to-one mapping of one field onto the other that preserves multiplication and addition. Thus we may speak of *the* finite field with q elements, usually denoted by $\mathbf{F}_q$ or GF(q).

Field of functions

Given a geometric object X, one can consider functions on such objects. Adding and multiplying them pointwise, i.e., $(f \cdot g)(x) = f(x) \cdot g(x)$ this leads to a field. However, due to the presence of possible zeros, i.e., points $x \in X$ where $f(x) = 0$, one has to take poles into account, i.e., formally allowing $f(x) = \infty$.

If X is an algebraic variety over F, then the rational functions $V \to F$, i.e., functions defined almost everywhere, form a field, the function field of V. Likewise, if X is a Riemann surface, then the meromorphic functions $S \to \mathbf{C}$ form a field. Under certain circumstances, namely when S is compact, S can be reconstructed from this field.

Local and global fields

Another important distinction in the realm of fields, especially with regard to number theory, are local fields and global fields. Local fields are completions of global fields at a given place. For example, **Q** is a global field, and the attached local fields are $\mathbf{Q}_p$ and **R** (Ostrowski's theorem). Algebraic number fields and function fields over $\mathbf{F}_q$ are further global fields. Studying arithmetic questions in global fields may sometimes be done by looking at the corresponding questions locally – this technique is called local-global principle.

Some first theorems

- Every finite subgroup of the multiplicative group $F^\times$ is cyclic. This applies in particular to $\mathbf{F}_q^{\times}$, it is cyclic of order $q - 1$. In the introductory example, a **generator** of $\mathbf{F}_4^{\times}$ is the element A.
- From the point of view of algebraic geometry, fields are points, because the spectrum *Spec F* has only one point, corresponding to the 0-ideal. This entails that a commutative ring is a field if and only if it has no ideals except $\{0\}$ and itself. Equivalently, an integral domain is field if and only if its Krull dimension is 0.
- Isomorphism extension theorem

Constructing fields

Closure operations

Assuming the axiom of choice, for every field F, there exists a field F, called the algebraic closure of F, which contains F, is algebraic over F, which means that any element x of F satisfies a polynomial equation

$$f_n x^n + f_{n-1} x^{n-1} + \ldots + f_1 x + f_0 = 0, \text{ with coefficients } f_n, \ldots, f_0 \in F,$$

and is algebraically closed, i.e., any such polynomial does have at least one solution in F. The algebraic closure is unique up to isomorphism inducing the identity on F. However, in many circumstances in mathematics, it is not appropriate to treat F as being uniquely determined by F, since the isomorphism above is not itself unique. In these cases, one refers to such a F as *an* algebraic closure of F. A similar concept is the separable closure, containing all roots of separable polynomials, instead of all polynomials.

For example, if F=**Q**, the algebraic closure **Q** is also called *field of algebraic numbers*. The field of algebraic numbers is an example of an algebraically closed field of characteristic zero; as such it satisfies the same first-order sentences as the field of complex numbers **C**.

In general, all algebraic closures of a field are isomorphic. However, there is in general no preferable isomorphism between two closures. Likewise for separable closures.

Subfields and field extensions

A *subfield* is, informally, a small field contained in a bigger one. Formally, a subfield E of a field F is a subset containing 0 and 1, closed under the operations + , −, · and multiplicative inverses and with its own operations defined by restriction. For example, the real numbers contain several interesting subfields: the real algebraic numbers, the computable numbers.

The notion of field extension lies at the heart of field theory, and as such is crucial to many other algebraic domains. A field extension F / E is simply a field F and a subfield $E \subset F$. Constructing such a field extension F / E can be done by "adding new elements" or *adjoining elements* to the field E. For example, given a field E, the set $F = E(X)$ of rational functions, i.e., equivalence classes of expressions of the kind

$$\frac{p(X)}{q(X)},$$

where $p(X)$ and $q(X)$ are polynomials with coefficients in E, and q is not the zero polynomial, forms a field. This is the simplest example of a transcendental extension of E. If one allows formal power series (also called *Laurent series*) in both denominator and numerator, one also gets a field, denoted $E((X))$.

In the above two cases, one is just adding a new symbol, namely X, which does not interact with elements of E. The following construction is different in this respect. This idea will first be exemplified by **R** vs. **C**. As explained above, **C** is a extension of **R**. The essential new element of **C**, in comparison to **R** is the imaginary unit i. It satisfies $\mathrm{i}^2 = -1$, or equivalently

$$\mathrm{i}^2 + 1 = 0.$$

Yet equivalently phrased, i is a zero of the polynomial $p(X) = X^2 + 1$. For any field F, the ring of polynomials with coefficients in F is denoted by F[X]. The corresponding quotient $C = \mathbf{R}[X] / (X^2 + 1)$ contains all real numbers and a variable X. In C, however, the relation $X^2 + 1 = 0$ holds. In other words, the element $X \in C$ satisfies exactly the property that i does. Therefore, the abstractly constructed C is isomorphic to the field **C** of complex numbers.

The above construction generalises to any irreducible polynomial in the polynomial ring $E[X]$, i.e., a polynomial $p(X)$ that cannot be written as a product of non-constant polynomials. The quotient $F = E[X] / (p(X))$, where $(p(X))$ denotes the ideal generated by $p(X)$, is again a field.

Alternatively, constructing such field extensions can also be done, if a bigger container is already given. Suppose given a field E, and a field G containing E as a subfield, for example G could be the algebraic closure of E. Let x be an element of G not in E. Then there is a smallest subfield of G containing E and x, denoted $F = E(x)$ and called *field extension F / E generated by x in G*. Such extensions are also called *simple extensions*. Many extensions are of this type, see the primitive element theorem. For instance, $\mathbf{Q}(i)$ is the subfield of **C** consisting of all numbers of the form $a + bi$ where both a and b are rational numbers.

One distinguishes between extensions having various qualities. For example, an extension K of a field k is called *algebraic*, if every element of K is a root of some polynomial with coefficients in k. Otherwise, the extension is called *transcendental*. The aim of Galois theory is the study of *algebraic extensions* of a field.

Rings vs. fields

Adding multiplicative inverses to an integral domain R yields the field of fractions of R. For example, the field of fractions of the integers **Z** is just **Q**. Also, the field $F(X)$ is the quotient field of the ring of polynomials $F[X]$. Getting back the ring from the field is sometimes possible; see discrete valuation ring.

Another method to obtain a field from a commutative ring R is taking the quotient R / m, where m is any maximal ideal of R. The above construction of $F = E[X] / (p(X))$, is an example, because the irreducibility of the polynomial $p(X)$ is equivalent to the maximality of the ideal generated by this polynomial. Another example are the finite fields $\mathbf{F}_p = \mathbf{Z} / p\mathbf{Z}$.

Ultraproducts

If I is an index set, U is an ultrafilter on I, and F_i is a field for every i in I, the ultraproduct of the F_i with respect to U is a field.

For example, a non-principal ultraproduct of finite fields is a pseudo finite field, i.e., a PAC field having exactly one extension of any degree. This construction is important to the study of the elementary theory of finite fields.

Galois theory

Galois theory aims to study the algebraic extensions of a field by studying the symmetry in the arithmetic operations of addition and multiplication. The fundamental theorem of Galois theory shows that there is a strong relation between the structure of the symmetry group and the set of algebraic extensions.

In the case where F / E is a finite (Galois) extension, Galois theory studies the algebraic extensions of E that are subfields of F. Such fields are called intermediate extensions. Specifically, the Galois group of F over E, denoted $\mathrm{Gal}(F/E)$, is the group of field automorphisms of F that are trivial on E (i.e., the bijections $\sigma : F \to F$ that preserve addition and multiplication and that send elements of E to themselves), and the fundamental theorem of Galois theory states that there is a one-to-one correspondence between subgroups of $\mathrm{Gal}(F/E)$ and the set of intermediate extensions of the extension F/E. The theorem, in fact, gives an explicit correspondence and further properties.

To study all (separable) algebraic extensions of E at once, one must consider the absolute Galois group of E, defined as the Galois group of the separable closure, E^{sep}, of E over E (i.e., $\mathrm{Gal}(E^{\text{sep}}/E)$. It is possible that the degree of this extension is infinite (as in the case of $E = \mathbf{Q}$). It is thus necessary to have a notion of Galois group for an infinite algebraic extension. The Galois group in this case is obtained as a "limit" (specifically an inverse limit) of the Galois groups of the finite Galois extensions of E. In this way, it acquires a topology[6] . The fundamental theorem of Galois theory can be generalized to the case of infinite Galois extensions by taking into consideration the topology of the Galois group, and in the case of E^{sep}/E it states that there this a one-to-one correspondence between *closed* subgroups of $\mathrm{Gal}(E^{\text{sep}}/E)$ and the set of all separable algebraic extensions of E (technically, one only obtains those separable algebraic extensions of E that occur as subfields of the *chosen* separable closure E^{sep}, but since all separable closures of E are isomorphic, choosing a different separable closure would give the same Galois group and thus an "equivalent" set of algebraic extensions).

Generalizations

There are also proper classes with field structure, which are sometimes called **Fields**, with a capital F:

- The surreal numbers form a Field containing the reals, and would be a field except for the fact that they are a proper class, not a set. The set of all surreal numbers with birthday smaller than some inaccessible cardinal form a field.
- The nimbers form a Field. The set of nimbers with birthday smaller than 2^{2^n}, the nimbers with birthday smaller than any infinite cardinal are all examples of fields.

In a different direction, differential fields are fields equipped with a derivation. For example, the field **R**(*X*), together with the standard derivative of polynomials forms a differential field. These fields are central to differential Galois theory. Exponential fields, meanwhile, are fields equipped with an exponential function that provides a homomorphism between the additive and multiplicative groups within the field. The usual exponential function makes the real and complex numbers exponential fields, denoted $\mathbf{R}_{\exp}$ and $\mathbf{C}_{\exp}$ respectively.

Generalizing in a more categorical direction yields the field with one element and related objects.

Exponentiation

One does not in general study generalization of field with *three* binary operations. The familiar addition/subtraction, multiplication/division, exponentiation/root-extraction operations from the natural numbers to the reals, each built up in terms of iteration of the last, mean that generalizing exponentiation as a binary operation is tempting, but has generally not proven fruitful; instead, an exponential field assumes a unary exponential function from the additive group to the multiplicative group, not a partially defined binary function. Note that the exponential operation of a^b is neither associative nor commutative, nor has a unique inverse (± 2 are both square roots of 4, for instance), unlike addition and multiplication, and further is not defined for many pairs – for example, $(-1)^{1/2} = \sqrt{-1}$ does not define a single number. These all show that even for rational numbers exponentiation is not nearly as well-behaved as addition and multiplication, which is why one does not in general axiomatize exponentiation.

Applications

The concept of a field is of use, for example, in defining vectors and matrices, two structures in linear algebra whose components can be elements of an arbitrary field.

Finite fields are used in number theory, Galois theory and coding theory, and again algebraic extension is an important tool.

Fields of characteristic 2 are useful in computer science.

See also

- Glossary of field theory for more definitions in field theory.
- Ring
- Vector space
- Category of fields
- Vector spaces without fields

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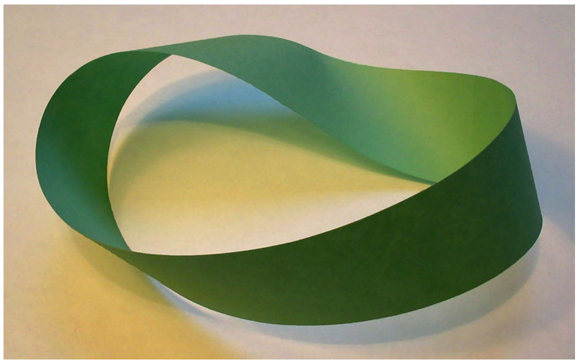
External links

- Field Theory Q&A (http://www.compsoc.nuigalway.ie/~pappasmurf/fields/index.php)
- Fields at ProvenMath (http://www.apronus.com/provenmath/fields.htm) definition and basic properties.
- Field (http://planetmath.org/?op=getobj&from=objects&id=355) on PlanetMath

Topology



A Möbius strip, an object with only one surface and one edge. Such shapes are an object of study in topology.

Topology (from the Greek τόπος, "place", and λόγος, "study") is a major area of mathematics concerned with spatial properties that are preserved under continuous deformations of objects, for example, deformations that involve stretching, but no tearing or gluing. It emerged through the development of concepts from geometry and set theory, such as space, dimension, and transformation.

Ideas that are now classified as topological were expressed as early as 1736, and toward the end of the 19th century, a distinct discipline developed, which was referred to in Latin as the *geometria situs* ("geometry of place") or *analysis situs* (Greek-Latin for "picking apart of place"), and which later acquired the modern name of topology. By the middle of the 20th century, topology had become an important area of study within mathematics.

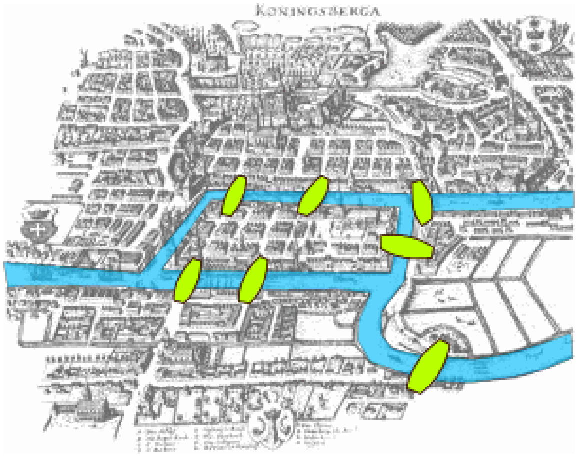
The word *topology* is used both for the mathematical discipline and for a family of sets with certain properties that are used to define a topological space, a basic object of topology. Of particular importance are *homeomorphisms*, which can be defined as continuous functions with a continuous inverse. For instance, the function $y = x^3$ is a homeomorphism of the real line.

Topology includes many subfields. The most basic and traditional division within topology is **point-set topology**, which establishes the foundational aspects of topology and investigates concepts inherent to topological spaces (basic examples include compactness and connectedness); **algebraic topology**, which generally tries to measure degrees of connectivity using algebraic constructs such as homotopy groups and homology; and **geometric**

topology, which primarily studies manifolds and their embeddings (placements) in other manifolds. Some of the most active areas, such as low dimensional topology and graph theory, do not fit neatly in this division.

See also: topology glossary for definitions of some of the terms used in topology and topological space for a more technical treatment of the subject.

History



The Seven Bridges of Königsberg is a famous problem solved by Euler.

Topology began with the investigation of certain questions in geometry. Euler's 1736 paper on *Seven Bridges of Königsberg* is regarded as one of the first academic treatises in modern topology.

The term "Topologie" was introduced in German in 1847 by Johann Benedict Listing in *Vorstudien zur Topologie*, Vandenhoeck und Ruprecht, Göttingen, pp. 67, 1848, who had used the word for ten years in correspondence before its first appearance in print. "Topology," its English form, was introduced in 1883 in the journal *Nature* to distinguish "qualitative geometry from the ordinary geometry in which quantitative relations chiefly are treated". The term **topologist** in the sense of a specialist in topology was used in 1905 in the magazine *Spectator*. However, none of these uses corresponds exactly to the modern definition of topology.

Modern topology depends strongly on the ideas of set theory, developed by Georg Cantor in the later part of the 19th century. Cantor, in addition to setting down the basic ideas of set theory, considered point sets in Euclidean space, as part of his study of Fourier series.

Henri Poincaré published *Analysis Situs* in 1895, introducing the concepts of homotopy and homology, which are now considered part of algebraic topology.

Maurice Fréchet, unifying the work on function spaces of Cantor, Volterra, Arzelà, Hadamard, Ascoli, and others, introduced the metric space in 1906. A metric space is now considered a special case of a general topological space. In 1914, Felix Hausdorff coined the term "topological space" and gave the definition for what is now called a Hausdorff space. In current usage, a topological space is a slight generalization of Hausdorff spaces, given in 1922 by Kazimierz Kuratowski.

For further developments, see point-set topology and algebraic topology.

Elementary introduction

Topology, as a branch of mathematics, can be formally defined as "the study of qualitative properties of certain objects (called topological spaces) that are invariant under certain kind of transformations (called continuous maps), especially those properties that are invariant under a certain kind of equivalence (called homeomorphism)."

The term *topology* is also used to refer to a structure imposed upon a set X, a structure which essentially 'characterizes' the set X as a topological space by taking proper care of properties such as convergence, connectedness and continuity, upon transformation.

Topological spaces show up naturally in almost every branch of mathematics. This has made topology one of the great unifying ideas of mathematics.

The motivating insight behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are put together. For example, the square and the circle have many properties in common: they are both one dimensional objects (from a topological point of view) and both separate the plane into two parts, the part inside and the part outside.

One of the first papers in topology was the demonstration, by Leonhard Euler, that it was impossible to find a route through the town of Königsberg (now Kaliningrad) that would cross each of its seven bridges exactly once. This result did not depend on the lengths of the bridges, nor on their distance from one another, but only on connectivity properties: which bridges are connected to which islands or riverbanks. This problem, the *Seven Bridges of Königsberg*, is now a famous problem in introductory mathematics, and led to the branch of mathematics known as graph theory.

Similarly, the hairy ball theorem of algebraic topology says that "one cannot comb the hair flat on a hairy ball without creating a cowlick." This fact is immediately convincing to most people, even though they might not recognize the more formal statement of the theorem, that there is no nonvanishing continuous tangent vector field on the sphere. As with the *Bridges of Königsberg*, the result does not depend on the exact shape of the sphere; it applies to pear shapes and in fact any kind of smooth blob, as long as it has no holes.

A continuous deformation (homeomorphism) of a coffee cup into a doughnut (torus) and back.

In order to deal with these problems that do not rely on the exact shape of the objects, one must be clear about just what properties these problems *do* rely on. From this need arises the notion of homeomorphism. The impossibility of crossing each bridge just once applies to any arrangement of bridges homeomorphic to those in Königsberg, and the hairy ball theorem applies to any space homeomorphic to a sphere.

Intuitively two spaces are homeomorphic if one can be deformed into the other without cutting or gluing. A traditional joke is that a topologist can't distinguish a coffee mug from a doughnut, since a sufficiently pliable doughnut could be reshaped to the form of a coffee cup by creating a dimple and progressively enlarging it, while shrinking the hole into a handle. A precise definition of homeomorphic, involving a continuous function with a continuous inverse, is necessarily more technical.

Homeomorphism can be considered the most basic *topological equivalence*. Another is homotopy equivalence. This is harder to describe without getting technical, but the essential notion is that two objects are homotopy equivalent if they both result from "squishing" some larger object.

Equivalence classes of the English alphabet in uppercase sans-serif font (Myriad); left - homeomorphism, right - homotopy equivalence

{A, R} {B} {C, G, I, J, L, M, N, S, U, V, W, Z} {D, O} {E, F, T, Y} {H, K} {P, Q} {X}	{A, R, D, O, P, Q} {B} {C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z}

An introductory exercise is to classify the uppercase letters of the English alphabet according to homeomorphism and homotopy equivalence. The result depends partially on the font used. The figures use a sans-serif font named Myriad. Notice that homotopy equivalence is a rougher relationship than homeomorphism; a homotopy equivalence class can contain several of the homeomorphism classes. The simple case of homotopy equivalence described above can be used here to show two letters are homotopy equivalent, e.g. O fits inside P and the tail of the P can be squished to the "hole" part.

Thus, the homeomorphism classes are: one hole two tails, two holes no tail, no holes, one hole no tail, no holes three tails, a bar with four tails (the "bar" on the *K* is almost too short to see), one hole one tail, and no holes four tails.

The homotopy classes are larger, because the tails can be squished down to a point. The homotopy classes are: one hole, two holes, and no holes.

To be sure we have classified the letters correctly, we not only need to show that two letters in the same class are equivalent, but that two letters in different classes are not equivalent. In the case of homeomorphism, this can be done by suitably selecting points and showing their removal disconnects the letters differently. For example, X and Y are not homeomorphic because removing the center point of the X leaves four pieces; whatever point in Y corresponds to this point, its removal can leave at most three pieces. The case of homotopy equivalence is harder and requires a more elaborate argument showing an algebraic invariant, such as the fundamental group, is different on the supposedly differing classes.

Letter topology has some practical relevance in stencil typography. The font Braggadocio, for instance, has stencils that are made of one connected piece of material.

Mathematical definition

Let **X** be any set and let ***T*** be a family of subsets of **X**. Then ***T*** is a **topology** on **X** iff

1. Both the empty set and **X** are elements of ***T***.
2. Any union of arbitrarily many elements of ***T*** is an element of ***T***.
3. Any intersection of finitely many elements of ***T*** is an element of ***T***.

If ***T*** is a topology on **X**, then the pair (**X**, ***T***) is called a **topological space**, and the notation $\mathbf{X}_{\boldsymbol{T}}$ is used to denote a set **X** endowed with the particular topology ***T***.

The **open** sets in X are defined to be the members of T; note that in general not all subsets of **X** need be in ***T***. A subset of **X** is said to be closed if its complement is in ***T*** (i.e., its complement is open). A subset of **X** may be open, closed, both, or neither.

A function or map from one topological space to another is called **continuous** if the inverse image of any open set is open. If the function maps the real numbers to the real numbers (both spaces with the Standard Topology), then this definition of continuous is equivalent to the definition of continuous in calculus. If a continuous function is one-to-one and onto and if the inverse of the function is also continuous, then the function is called a homeomorphism and the domain of the function is said to be homeomorphic to the range. Another way of saying this is that the function has a natural extension to the topology. If two spaces are homeomorphic, they have identical topological properties, and are considered to be topologically the same. The cube and the sphere are homeomorphic, as are the coffee cup and the doughnut. But the circle is not homeomorphic to the doughnut.

Topology topics

Some theorems in general topology

- Every closed interval in **R** of finite length is compact. More is true: In $\mathbf{R}^n$, a set is compact if and only if it is closed and bounded. (See Heine-Borel theorem).
- Every continuous image of a compact space is compact.
- Tychonoff's theorem: The (arbitrary) product of compact spaces is compact.
- A compact subspace of a Hausdorff space is closed.
- Every continuous bijection from a compact space to a Hausdorff space is necessarily a homeomorphism.
- Every sequence of points in a compact metric space has a convergent subsequence.
- Every interval in **R** is connected.
- Every compact m-manifold can be embedded in some Euclidean space $\mathbf{R}^n$.
- The continuous image of a connected space is connected.
- A metric space is Hausdorff, also normal and paracompact.
- The metrization theorems provide necessary and sufficient conditions for a topology to come from a metric.
- The Tietze extension theorem: In a normal space, every continuous real-valued function defined on a closed subspace can be extended to a continuous map defined on the whole space.
- Any open subspace of a Baire space is itself a Baire space.
- The Baire category theorem: If *X* is a complete metric space or a locally compact Hausdorff space, then the interior of every union of countably many nowhere dense sets is empty.
- On a paracompact Hausdorff space every open cover admits a partition of unity subordinate to the cover.
- Every path-connected, locally path-connected and semi-locally simply connected space has a universal cover.

General topology also has some surprising connections to other areas of mathematics. For example:

- In number theory, Furstenberg's proof of the infinitude of primes.

See also some counter-intuitive theorems, e.g. the Banach-Tarski one.

Some useful notions from algebraic topology

See also list of algebraic topology topics.

- Homology and cohomology: Betti numbers, Euler characteristic, degree of a continuous mapping.
- Operations: cup product, Massey product
- Intuitively-attractive applications: Brouwer fixed-point theorem, Hairy ball theorem, Borsuk-Ulam theorem, Ham sandwich theorem.
- Homotopy groups (including the fundamental group).
- Chern classes, Stiefel-Whitney classes, Pontryagin classes.

Generalizations

Occasionally, one needs to use the tools of topology but a "set of points" is not available. In pointless topology one considers instead the lattice of open sets as the basic notion of the theory, while Grothendieck topologies are certain structures defined on arbitrary categories which allow the definition of sheaves on those categories, and with that the definition of quite general cohomology theories.

Topology in art and literature

- Some M. C. Escher works illustrate topological concepts, such as Möbius strips and non-orientable spaces.

See also

- List of algebraic topology topics
- List of general topology topics
- List of geometric topology topics
- Publications in topology
- List of topology topics
- Topology glossary

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External links

- Elementary Topology: A First Course [1] Viro, Ivanov, Netsvetaev, Kharlamov
- Topology [2] at the Open Directory Project
- The Topological Zoo [3] at The Geometry Center
- Topology Atlas [4]
- Topology Course Lecture Notes [5] Aisling McCluskey and Brian McMaster, Topology Atlas
- Topology Glossary [6]
- Moscow 1935: Topology moving towards America [7], a historical essay by Hassler Whitney.

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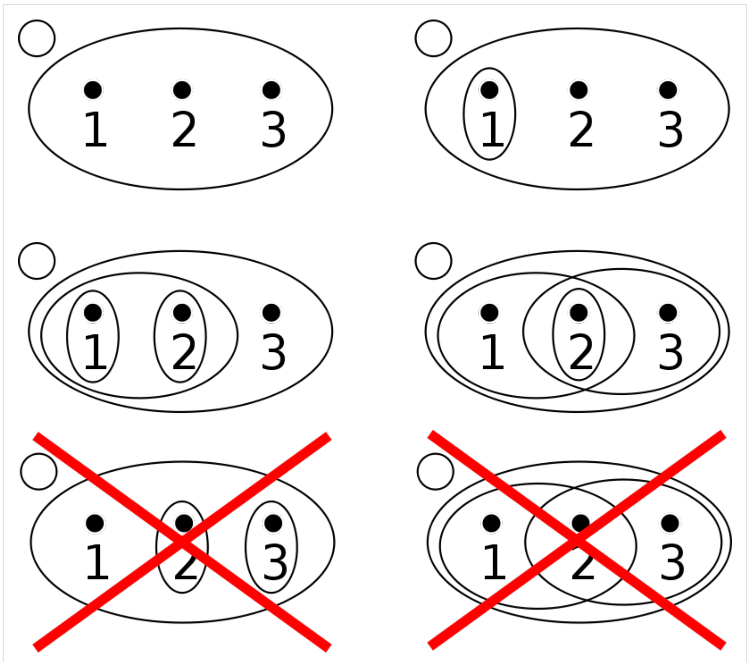
[2] http://www.dmoz.org/Science/Math/Topology/
[3] http://www.geom.uiuc.edu/zoo/
[4] http://at.yorku.ca/topology/
[5] http://at.yorku.ca/i/a/a/b/23.htm
[6] http://www.ornl.gov/sci/ortep/topology/defs.txt
[7] http://www.ams.org/online_bks/hmath1/hmath1-whitney10.pdf

Topological space

Topological spaces are mathematical structures that allow the formal definition of concepts such as convergence, connectedness, and continuity. They appear in virtually every branch of modern mathematics and are a central unifying notion. The branch of mathematics that studies topological spaces in their own right is called topology.



Four examples and two non-examples of topologies on the three-point set {1,2,3}. The bottom-left example is not a topology because the union {2,3} of {2} and {3} is missing; the bottom-right example is not a topology because the intersection {2} of {1,2} and {2,3} is missing.

Definition

A **topological space** is a set X together with τ, a collection of subsets of X, satisfying the following axioms:

1. The empty set and X are in τ.
2. The union of any collection of sets in τ is also in τ.
3. The intersection of any finite collection of sets in τ is also in τ.

The collection τ is called a **topology** on X. The elements of X are usually called *points*, though they can be any mathematical objects. A topological space in which the *points* are functions is called a function space. The sets in τ are the open sets, and their complements in X are called closed sets. A set may be neither closed nor open, either closed or open, or both. A set that is both closed and open is called a clopen set.

Examples

1. $X = \{1, 2, 3, 4\}$ and collection $\tau =$ of only the two subsets of X required by the axioms form a topology, the trivial topology.
2. $X = \{1, 2, 3, 4\}$ and collection $\tau =$ of six subsets of X form another topology.
3. $X = \{1, 2, 3, 4\}$ and collection $\tau = P(X)$ (the power set of X) form a third topology, the discrete topology.
4. $X = \mathbf{Z}$, the set of integers, and collection τ equal to all finite subsets of the integers plus $\mathbf{Z}$ itself is *not* a topology, because (for example) the union of all finite sets not containing zero is infinite but is not all of $\mathbf{Z}$, and so is not in τ.

Equivalent definitions

There are many other equivalent ways to define a topological space. (In other words, each of the following defines a category equivalent to the category of topological spaces above.) For example, using de Morgan's laws, the axioms defining open sets above become axioms defining closed sets:

1. The empty set and *X* are closed.
2. The intersection of any collection of closed sets is also closed.
3. The union of any pair of closed sets is also closed.

Using these axioms, another way to define a topological space is as a set *X* together with a collection τ of subsets of *X* satisfying the following axioms:

1. The empty set and *X* are in τ.
2. The intersection of any collection of sets in τ is also in τ.
3. The union of any pair of sets in τ is also in τ.

Under this definition, the sets in the topology τ are the closed sets, and their complements in *X* are the open sets.

Another way to define a topological space is by using the Kuratowski closure axioms, which define the closed sets as the fixed points of an operator on the power set of X.

A neighbourhood of a point *x* is any set that contains an open set containing *x*. The *neighbourhood system* at *x* consists of all neighbourhoods of *x*. A topology can be determined by a set of axioms concerning all neighbourhood systems.

A net is a generalisation of the concept of sequence. A topology is completely determined if for every net in *X* the set of its accumulation points is specified.

Comparison of topologies

A variety of topologies can be placed on a set to form a topological space. When every set in a topology τ_1 is also in a topology τ_2, we say that τ_2 is *finer* than τ_1, and τ_1 is *coarser* than τ_2. A proof which relies only on the existence of certain open sets will also hold for any finer topology, and similarly a proof that relies only on certain sets not being open applies to any coarser topology. The terms *larger* and *smaller* are sometimes used in place of finer and coarser, respectively. The terms *stronger* and *weaker* are also used in the literature, but with little agreement on the meaning, so one should always be sure of an author's convention when reading.

The collection of all topologies on a given fixed set *X* forms a complete lattice: if *F* = { τ_α : α in A} is a collection of topologies on *X*, then the meet of *F* is the intersection of *F*, and the join of *F* is the meet of the collection of all topologies on *X* which contain every member of *F*.

Continuous functions

A function between topological spaces is said to be **continuous** if the inverse image of every open set is open. This is an attempt to capture the intuition that there are no "breaks" or "separations" in the function. A homeomorphism is a bijection that is continuous and whose inverse is also continuous. Two spaces are said to be *homeomorphic* if there exists a homeomorphism between them. From the standpoint of topology, homeomorphic spaces are essentially identical.

In category theory, **Top**, the category of topological spaces with topological spaces as objects and continuous functions as morphisms is one of the fundamental categories in mathematics. The attempt to classify the objects of this category (up to homeomorphism) by invariants has motivated and generated entire areas of research, such as homotopy theory, homology theory, and K-theory, to name just a few.

Examples of topological spaces

A given set may have many different topologies. If a set is given a different topology, it is viewed as a different topological space. Any set can be given the discrete topology in which every subset is open. The only convergent sequences or nets in this topology are those that are eventually constant. Also, any set can be given the trivial topology (also called the indiscrete topology), in which only the empty set and the whole space are open. Every sequence and net in this topology converges to every point of the space. This example shows that in general topological spaces, limits of sequences need not be unique. However, often topological spaces are required to be Hausdorff spaces where limit points are unique.

There are many ways of defining a topology on **R**, the set of real numbers. The standard topology on **R** is generated by the open intervals. The set of all open intervals forms a base or basis for the topology, meaning that every open set is a union of some collection of sets from the base. In particular, this means that a set is open if there exists an open interval of non zero radius about every point in the set. More generally, the Euclidean spaces $\mathbf{R}^n$ can be given a topology. In the usual topology on $\mathbf{R}^n$ the basic open sets are the open balls. Similarly, **C** and $\mathbf{C}^n$ have a standard topology in which the basic open sets are open balls.

Every metric space can be given a metric topology, in which the basic open sets are open balls defined by the metric. This is the standard topology on any normed vector space. On a finite-dimensional vector space this topology is the same for all norms.

Many sets of operators in functional analysis are endowed with topologies that are defined by specifying when a particular sequence of functions converges to the zero function.

Any local field has a topology native to it, and this can be extended to vector spaces over that field.

Every manifold has a natural topology since it is locally Euclidean. Similarly, every simplex and every simplicial complex inherits a natural topology from $\mathbf{R}^n$.

The Zariski topology is defined algebraically on the spectrum of a ring or an algebraic variety. On $\mathbf{R}^n$ or $\mathbf{C}^n$, the closed sets of the Zariski topology are the solution sets of systems of polynomial equations.

A linear graph has a natural topology that generalises many of the geometric aspects of graphs with vertices and edges.

The Sierpiński space is the simplest non-discrete topological space. It has important relations to the theory of computation and semantics.

There exist numerous topologies on any given finite set. Such spaces are called finite topological spaces. Finite spaces are sometimes used to provide examples or counterexamples to conjectures about topological spaces in general.

Any set can be given the cofinite topology in which the open sets are the empty set and the sets whose complement is finite. This is the smallest T_1 topology on any infinite set.

Any set can be given the cocountable topology, in which a set is defined to be open if it is either empty or its complement is countable. When the set is uncountable, this topology serves as a counterexample in many situations.

The real line can also be given the lower limit topology. Here, the basic open sets are the half open intervals $[a, b)$. This topology on **R** is strictly finer than the Euclidean topology defined above; a sequence converges to a point in this topology if and only if it converges from above in the Euclidean topology. This example shows that a set may have many distinct topologies defined on it.

If Γ is an ordinal number, then the set $\Gamma = [0, \Gamma)$ may be endowed with the order topology generated by the intervals (a, b), $[0, b)$ and (a, Γ) where a and b are elements of Γ.

Topological constructions

Every subset of a topological space can be given the subspace topology in which the open sets are the intersections of the open sets of the larger space with the subset. For any indexed family of topological spaces, the product can be given the product topology, which is generated by the inverse images of open sets of the factors under the projection mappings. For example, in finite products, a basis for the product topology consists of all products of open sets. For infinite products, there is the additional requirement that in a basic open set, all but finitely many of its projections are the entire space.

A quotient space is defined as follows: if X is a topological space and Y is a set, and if $f : X \to Y$ is a surjective function, then the quotient topology on Y is the collection of subsets of Y that have open inverse images under f. In other words, the quotient topology is the finest topology on Y for which f is continuous. A common example of a quotient topology is when an equivalence relation is defined on the topological space X. The map f is then the natural projection onto the set of equivalence classes.

The **Vietoris topology** on the set of all non-empty subsets of a topological space X, named for Leopold Vietoris, is generated by the following basis: for every n-tuple $U_1, ..., U_n$ of open sets in X, we construct a basis set consisting of all subsets of the union of the U_i which have non-empty intersection with each U_i.

Classification of topological spaces

Topological spaces can be broadly classified, up to homeomorphism, by their topological properties. A topological property is a property of spaces that is invariant under homeomorphisms. To prove that two spaces are not homeomorphic it is sufficient to find a topological property which is not shared by them. Examples of such properties include connectedness, compactness, and various separation axioms.

See the article on *topological properties* for more details and examples.

Topological spaces with algebraic structure

For any algebraic objects we can introduce the discrete topology, under which the algebraic operations are continuous functions. For any such structure which is not finite, we often have a natural topology which is compatible with the algebraic operations in the sense that the algebraic operations are still continuous. This leads to concepts such as topological groups, topological vector spaces, topological rings and local fields.

Topological spaces with order structure

- **Spectral**. A space is spectral if and only if it is the prime spectrum of a ring (Hochster theorem).
- **Specialization preorder**. In a space the **specialization** (or **canonical**) **preorder** is defined by $x \le y$ if and only if $\mathrm{cl}\{x\} \subseteq \mathrm{cl}\{y\}$.

Specializations and generalizations

The following spaces and algebras are either more specialized or more general than the topological spaces discussed above.

- Proximity spaces provide a notion of closeness of two sets.
- Metric spaces embody a metric, a precise notion of distance between points.
- Uniform spaces axiomatize ordering the distance between distinct points.
- Cauchy spaces axiomatize the ability to test whether a net is Cauchy. Cauchy spaces provide a general setting for studying completions.
- Convergence spaces capture some of the features of convergence of filters.
- σ-algebras build on the notion of measurable sets.

See also

- Kolmogorov space (T_0)
- accessible/Fréchet space (T_1)
- Hausdorff space (T_2)
- completely Hausdorff space and Urysohn space ($T_{2\frac{1}{2}}$)
- regular space and regular Hausdorff space (T_3)
- Tychonoff space and completely regular space ($T_{3\frac{1}{2}}$)
- normal Hausdorff space (T_4)
- completely normal Hausdorff space (T_5)
- perfectly normal Hausdorff space (T_6)
- Space (mathematics)

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External links

- Topological space [1] on PlanetMath

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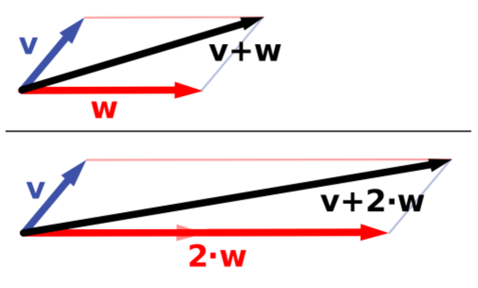
Vector space



Vector addition and scalar multiplication: a vector **v** (blue) is added to another vector **w** (red, upper illustration). Below, **w** is stretched by a factor of 2, yielding the sum **v** + 2·**w**.

A **vector space** is a mathematical structure formed by a collection of vectors: objects that may be added together and multiplied ("scaled") by numbers, called *scalars* in this context. Scalars are often taken to be real numbers, but one may also consider vector spaces with scalar multiplication by complex numbers, rational numbers, or even more general fields instead. The operations of vector addition and scalar multiplication have to satisfy certain requirements, called *axioms*, listed below. An example of a vector space is that of Euclidean vectors which are often used to represent physical quantities such as forces: any two forces (of the same type) can be added to yield a third, and the multiplication of a force vector by a real factor is another force vector. In the same vein, but in more geometric parlance, vectors representing displacements in the plane or in three-dimensional space also form vector spaces.

Vector spaces are the subject of linear algebra and are well understood from this point of view, since vector spaces are characterized by their dimension, which, roughly speaking, specifies the number of independent directions in the space. The theory is further enhanced by introducing on a vector space some additional structure, such as a norm or inner product. Such spaces arise naturally in mathematical analysis, mainly in the guise of infinite-dimensional function spaces whose vectors are functions. Analytical problems call for the ability to decide if a sequence of vectors converges to a given vector. This is accomplished by considering vector spaces with additional data, mostly spaces endowed with a suitable topology, thus allowing the consideration of proximity and continuity issues. These topological vector spaces, in particular Banach spaces and Hilbert spaces, have a richer theory.

Historically, the first ideas leading to vector spaces can be traced back as far as 17th century's analytic geometry, matrices, systems of linear equations, and Euclidean vectors. The modern, more abstract treatment, first formulated by Giuseppe Peano in the late 19th century, encompasses more general objects than Euclidean space, but much of the theory can be seen as an extension of classical geometric ideas like lines, planes and their higher-dimensional analogs.

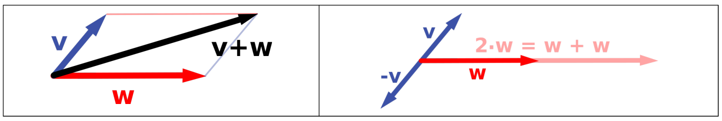
Today, vector spaces are applied throughout mathematics, science and engineering. They are the appropriate linear-algebraic notion to deal with systems of linear equations; offer a framework for Fourier expansion, which is employed in image compression routines; or provide an environment that can be used for solution techniques for partial differential equations. Furthermore, vector spaces furnish an abstract, coordinate-free way of dealing with geometrical and physical objects such as tensors. This in turn allows the examination of local properties of manifolds by linearization techniques. Vector spaces may be generalized in several directions, leading to more advanced notions in geometry and abstract algebra.

Introduction and definition

The concept of vector space relies on the idea of *vectors*. A first example of vectors are arrows in a fixed plane, starting at one fixed point. Such vectors are called Euclidean vectors and can be used to describe physical forces or velocities or further entities having both a magnitude and a direction. In general, the term vector is used for objects on which two operations can be exerted. The concrete nature of these operations depends on the type of vector under consideration, and can often be described by different means, e.g. geometric or algebraic. In view of the algebraic ideas behind these concepts explained below, the two operations are called *vector addition* and *scalar multiplication*.

Vector addition means that two vectors **v** and **w** can be "added" to yield the sum **v** + **w**, another vector. The sum of two arrow vectors is calculated by constructing the parallelogram two of whose sides are the given vectors **v** and **w**. The sum of the two is given by the diagonal arrow of the parallelogram, starting at the common point of the two vectors (left-most image below).

Scalar multiplication combines a number—also called *scalar*—r and a vector **v**. In the example, a vector represented by an arrow is multiplied by a scalar by dilating or shrinking the arrow accordingly: if $r = 2$ ($r = 1/4$), the resulting vector $r \cdot \mathbf{w}$ has the same direction as **w**, but is stretched to the double length (shrunk to a fourth of the length, respectively) of **w** (right image below). Equivalently $2 \cdot \mathbf{w}$ is the sum $\mathbf{w} + \mathbf{w}$. In addition, for negative factors, the direction of the arrow is swapped: $(-1) \cdot \mathbf{v} = -\mathbf{v}$ has the opposite direction and the same length as **v** (blue vector in the right image).



Another example of vectors is provided by pairs of real numbers x and y, denoted (x, y). (The order of the components x and y is significant, so such a pair is also called an ordered pair.) These pairs form vectors, by defining vector addition and scalar multiplication componentwise, i.e.

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$r \cdot (x, y) = (rx, ry).$$

Definition

Incorporating these two and many more examples in one notion of vector space is achieved via an abstract algebraic definition that disregards the concrete nature of the particular type of vectors. However, essential properties of vector addition and scalar multiplication present in the examples above are required to hold in any vector space. For example, in the algebraic example of vectors as pairs above, the result of addition does not depend on the order of the summands:

$$(x_\mathbf{v}, y_\mathbf{v}) + (x_\mathbf{w}, y_\mathbf{w}) = (x_\mathbf{w}, y_\mathbf{w}) + (x_\mathbf{v}, y_\mathbf{v}),$$

Likewise, in the geometric example of vectors using arrows, $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$, since the parallelogram defining the sum of the vectors is independent of the order of the vectors.

To reach utmost generality, the definition of a vector space relies on the notion of a field F. A field is, essentially, a set of numbers possessing addition, subtraction, multiplication and division operations.[1] Many vector spaces encountered in mathematics and sciences use the field of real numbers, but rational or complex numbers and other fields are also important. The underlying field F is fixed throughout and is specified by speaking of *F-vector spaces* or *vector spaces over F*. If F is **R** or **C**, the field of real and complex numbers, respectively, the denominations *real* and *complex vector spaces* are also common. The elements of F are called scalars.

A *vector space* is a set V together with two binary operations, operations that combine two entities to yield a third, called vector addition and scalar multiplication. The elements of V are called *vectors* and are denoted in boldface.[2] The sum of two vectors is denoted $\mathbf{v} + \mathbf{w}$, the product of a scalar a and a vector **v** is denoted $a \cdot \mathbf{v}$ or $a\mathbf{v}$.

To qualify as a vector space, addition and multiplication have to adhere to a number of requirements called axioms. They generalize properties of the vectors introduced above.[3] In the list below, let **u**, **v**, **w** be arbitrary vectors in V, and a, b be scalars in F.

Axiom	Signification
Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
Commutativity of addition	$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
Identity element of addition	There exists an element $\mathbf{0} \in V$, called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of addition	For all $\mathbf{v} \in$ V, there exists an element $\mathbf{w} \in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$. The additive inverse is denoted $-\mathbf{v}$.
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$.
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$ [4]
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F.

These axioms entail that subtraction of two vectors and division by a (non-zero) scalar can be performed via

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}),$$

$$\mathbf{v} / a = (1 / a) \cdot \mathbf{v}.$$

In contrast to the intuition stemming from vectors in the plane and higher-dimensional cases, there is, in general vector spaces, no notion of nearness, angles or distances. To deal with such matters, particular types of vector spaces are introduced; see below.

Alternative formulations and elementary consequences

The requirement that vector addition and scalar multiplication be binary operations includes (by definition of binary operations) a property called closure: that $\mathbf{u} + \mathbf{v}$ and $a\mathbf{v}$ are in V for all a in F, and $\mathbf{u}$, $\mathbf{v}$ in V. Some older sources mention these properties as separate axioms.[5]

In the parlance of abstract algebra, the first four axioms can be subsumed by requiring the set of vectors to be an abelian group under addition. The remaining axioms give this group an F-module structure. In other words there is a ring homomorphism f from the field F into the endomorphism ring of the group of vectors. Then scalar multiplication $a\mathbf{v}$ is defined as $(f(a))(\mathbf{v})$.[6]

There are a number of direct consequences of the vector space axioms. Some of them derive from elementary group theory, applied to the additive group of vectors: for example the zero vector $\mathbf{0}$ of V and the additive inverse $-\mathbf{v}$ of any vector $\mathbf{v}$ are unique. Other properties follow from the distributive law, for example $a\mathbf{v}$ equals $\mathbf{0}$ if and only if a equals 0 or $\mathbf{v}$ equals $\mathbf{0}$.

History

Vector spaces stem from affine geometry, via the introduction of coordinates in the plane or three-dimensional space. Around 1636, Descartes and Fermat founded analytic geometry by identifying solutions to an equation of two variables with points on a plane curve.[7] To achieve geometric solutions without using coordinates, Bolzano introduced, in 1804, certain operations on points, lines and planes, which are predecessors of vectors.[8] This work was made use of in the conception of barycentric coordinates by Möbius in 1827.[9] The foundation of the definition of vectors was Bellavitis' notion of the bipoint, an oriented segment one of whose ends is the origin and the other one a target. Vectors were reconsidered with the presentation of complex numbers by Argand and Hamilton and the inception of quaternions by the latter.[10] They are elements in $\mathbf{R}^2$ and $\mathbf{R}^4$; treating them using linear combinations goes back to Laguerre in 1867, who also defined systems of linear equations.

In 1857, Cayley introduced the matrix notation which allows for a harmonization and simplification of linear maps. Around the same time, Grassmann studied the barycentric calculus initiated by Möbius. He envisaged sets of abstract objects endowed with operations.[11] In his work, the concepts of linear independence and dimension, as well as scalar products are present. Actually Grassmann's 1844 work exceeds the framework of vector spaces, since his considering multiplication, too, led him to what are today called algebras. Peano was the first to give the modern definition of vector spaces and linear maps in 1888.[12]

An important development of vector spaces is due to the construction of function spaces by Lebesgue. This was later formalized by Banach and Hilbert, around 1920.[13] At that time, algebra and the new field of functional analysis began to interact, notably with key concepts such as spaces of p-integrable functions and Hilbert spaces.[14] Vector spaces, including infinite-dimensional ones, then became a firmly established notion, and many mathematical branches started making use of this concept.

Examples

Coordinate and function spaces

The first example of a vector space over a field F is the field itself, equipped with its standard addition and multiplication. This is the case $n = 1$ of a vector space usually denoted F^n, known as the *coordinate space* whose elements are n-tuples (sequences of length n):

$(a_1, a_2, \ldots, a_n)$, where the a_i are elements of F.[15]

The case $F = \mathbf{R}$ and $n = 2$ was discussed in the introduction above. Infinite coordinate sequences, and more generally functions from any fixed set Ω to a field F also form vector spaces, by performing addition and scalar multiplication pointwise. That is, the sum of two functions f and g is given by

$$(f + g)(w) = f(w) + g(w)$$

and similarly for multiplication. Such function spaces occur in many geometric situations, when Ω is the real line or an interval, or other subsets of $\mathbf{R}^n$. Many notions in topology and analysis, such as continuity, integrability or differentiability are well-behaved with respect to linearity: sums and scalar multiples of functions possessing such a property still have that property.[16] Therefore, the set of such functions are vector spaces. They are studied in greater detail using the methods of functional analysis, see below. Algebraic constraints also yield vector spaces: the vector space $F[x]$ is given by polynomial functions:

$f(x) = r_0 + r_1 x + \ldots + r_{n-1} x^{n-1} + r_n x^n$, where the coefficients $r_0, \ldots, r_n$ are in F.[17]

Linear equations

Systems of homogeneous linear equations are closely tied to vector spaces.[18] For example, the solutions of

$$\begin{alignedat}{4} a &{}+{}& 3b &{}+{}& c &= 0 \\ 4a &{}+{}& 2b &{}+{}& 2c &= 0 \end{alignedat}$$

are given by triples with arbitrary a, $b = a/2$, and $c = -5a/2$. They form a vector space: sums and scalar multiples of such triples still satisfy the same ratios of the three variables; thus they are solutions, too. Matrices can be used to condense multiple linear equations as above into one vector equation, namely

$$A\mathbf{x} = \mathbf{0},$$

where $A = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 2 \end{bmatrix}$ is the matrix containing the coefficients of the given equations, $\mathbf{x}$ is the vector (a, b, c), $A\mathbf{x}$ denotes the matrix product and $\mathbf{0} = (0, 0)$ is the zero vector. In a similar vein, the solutions of homogeneous *linear differential equations* form vector spaces. For example

$$f''(x) + 2f'(x) + f(x) = 0$$

yields $f(x) = a\,e^{-x} + bx\,e^{-x}$, where a and b are arbitrary constants, and e^x is the natural exponential function.

Field extensions

Field extensions F / E ("F over E") provide another class of examples of vector spaces, particularly in algebra and algebraic number theory: a field F containing a smaller field E becomes an E-vector space, by the given multiplication and addition operations of F.[19] For example the complex numbers are a vector space over $\mathbf{R}$. A particularly interesting type of field extension in number theory is $\mathbf{Q}(\alpha)$, the extension of the rational numbers $\mathbf{Q}$ by a fixed complex number α. $\mathbf{Q}(\alpha)$ is the smallest field containing the rationals and a fixed complex number α. Its dimension as a vector space over $\mathbf{Q}$ depends on the choice of α.

Bases and dimension

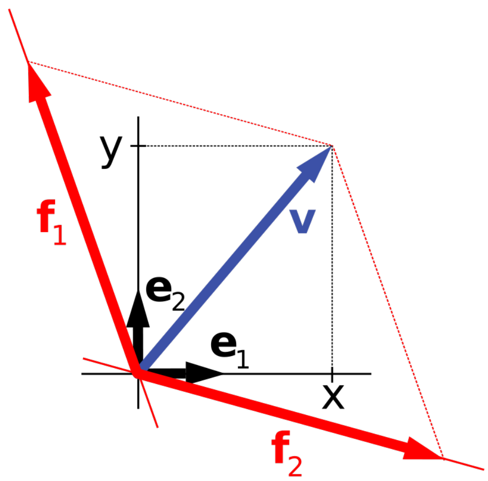
Bases reveal the structure of vector spaces in a concise way. A basis is defined as a (finite or infinite) set $B = \{\mathbf{v}_i\}_{i \in I}$ of vectors $\mathbf{v}_i$ indexed by some index set I that *spans* the whole space, and is minimal with this property. The former means that any vector $\mathbf{v}$ can be expressed as a finite sum (called *linear combination* of the basis elements)

$$\mathbf{v} = a_1\mathbf{v}_i1 + a_2\mathbf{v}_i2 + \ldots + a_n\mathbf{v}_in,$$

where the a_k are scalars and $\mathbf{v}_ik$ ($k = 1, \ldots, n$) elements of the basis B. Minimality, on the other hand, is made formal by requiring B to be *linearly independent*. A set of vectors is said to be linearly independent if none of its elements can be expressed as a linear combination of the remaining ones. Equivalently, an equation

$$a_1\mathbf{v}_i1 + a_2\mathbf{v}_i2 + \ldots + a_n\mathbf{v}_i\mathrm{n} = 0$$

can only hold if all scalars $a_1, \ldots, a_n$ equal zero. Linear independence ensures that the representation of any vector in terms of basis vectors, the existence of which is guaranteed by the requirement that the basis span V, is unique.[20] This is referred to as the coordinatized viewpoint of vector spaces, by viewing basis vectors as generalizations of coordinate vectors x, y, z in $\mathbf{R}^3$ and similarly in higher-dimensional cases.



A vector $\mathbf{v}$ in $\mathbf{R}^2$ (blue) expressed in terms of different bases: using the standard basis of $\mathbf{R}^2$ $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2$ (black), and using a different, non-orthogonal basis: $\mathbf{v} = \mathbf{f}_1 + \mathbf{f}_2$ (red).

The coordinate vectors $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, to $\mathbf{e}_n = (0, 0, \ldots, 0, 1)$, form basis of F^n, called the standard basis, since any vector $(x_1, x_2, \ldots, x_n)$ can be uniquely expressed as a linear combination of these vectors:

$$(x_1, x_2, \ldots, x_n) = x_1(1, 0, \ldots, 0) + x_2(0, 1, 0, \ldots, 0) + \ldots + x_n(0, \ldots, 0, 1) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n.$$

Every vector space has a basis. This follows from Zorn's lemma, an equivalent formulation of the axiom of choice.[21] Given the other axioms of Zermelo-Fraenkel set theory, the existence of bases is equivalent to the axiom of choice.[22] The ultrafilter lemma, which is weaker than the axiom of choice, implies that all bases of a given vector space have the same number of elements, or cardinality.[23] It is called the *dimension* of the vector space, denoted dim V. If the space is spanned by finitely many vectors, the above statements can be proven without such fundamental input from set theory.[24]

The dimension of the coordinate space F^n is n, by the basis exhibited above. The dimension of the polynomial ring $F[x]$ introduced above is countably infinite, a basis is given by $1, x, x^2, \ldots$ A fortiori, the dimension of more general function spaces, such as the space of functions on some (bounded or unbounded) interval, is infinite.[25] Under suitable regularity assumptions on the coefficients involved, the dimension of the solution space of a homogeneous ordinary differential equation equals the degree of the equation.[26] For example, the solution space above equation is generated by e^{-x} and xe^{-x}. These two functions are linearly independent over **R**, so the dimension of this space is two, as is the degree of the equation.

The dimension (or degree) of the field extension **Q**(α) over **Q** depends on α. If α satisfies some polynomial equation

$$q_n\alpha^n + q_{n-1}\alpha^{n-1} + \ldots + q_0 = 0, \text{ with rational coefficients } q_n, \ldots, q_0.$$

("α is algebraic"), the dimension is finite. More precisely, it equals the degree of the minimal polynomial having α as a root.[27] For example, the complex numbers **C** are a two-dimensional real vector space, generated by 1 and the imaginary unit i. The latter satisfies $i^2 + 1 = 0$, an equation of degree two. Thus, **C** is a two-dimensional **R**-vector space (and, as any field, one-dimensional as a vector space over itself, **C**). If α is not algebraic, the dimension of **Q**(α) over **Q** is infinite. For instance, for α = π there is no such equation, in other words π is transcendental.[28]

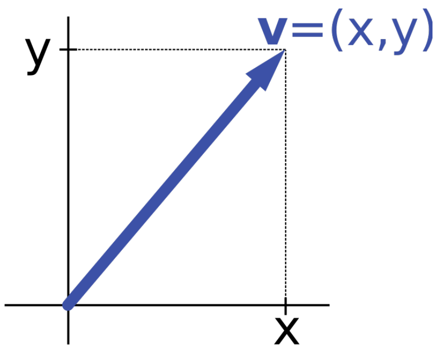
Linear maps and matrices

The relation of two vector spaces can be expressed by *linear map* or *linear transformation*. They are functions that reflect the vector space structure—i.e., they preserve sums and scalar multiplication:

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}) \text{ and } f(a \cdot \mathbf{x}) = a \cdot f(\mathbf{x}) \text{ for all } \mathbf{x} \text{ and } \mathbf{y} \text{ in } V, \text{ all } a \text{ in } F.$$ [29]

An *isomorphism* is a linear map $f : V \to W$ such that there exists an inverse map $g : W \to V$, which is a map such that the two possible compositions $f \circ g : W \to W$ and $g \circ f : V \to V$ are identity maps. Equivalently, f is both one-to-one (injective) and onto (surjective).[30] If there exists an isomorphism between V and W, the two spaces are said to be *isomorphic*; they are then essentially identical as vector spaces, since all identities holding in V are, via f, transported to similar ones in W, and vice versa via g.

For example, the vector spaces in the introduction are isomorphic: a planar arrow **v** departing at the origin of some (fixed) coordinate system can be expressed as an ordered pair by considering the x- and y-component of the arrow, as shown in the image at the right. Conversely, given a pair (x, y), the arrow going by x to the right (or to the left, if x is negative), and y up (down, if y is negative) turns back the arrow **v**.



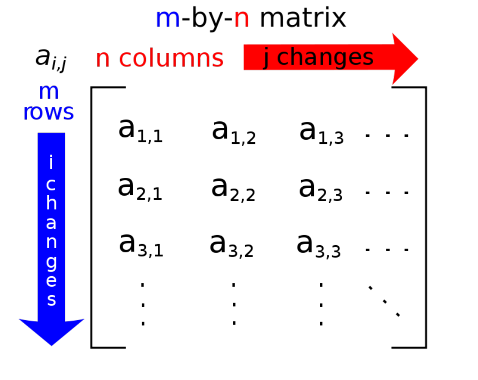
Describing an arrow vector **v** by its coordinates x and y yields an isomorphism of vector spaces.

Linear maps $V \to W$ between two fixed vector spaces form a vector space $\mathrm{Hom}_F(V, W)$, also denoted $\mathrm{L}(V, W)$.[31] The space of linear maps from V to F is called the *dual vector space*, denoted V^*.[32] Via the injective natural map $V \to V^{**}$, any vector space can be embedded into its *bidual*; the map is an isomorphism if and only if the space is finite-dimensional.[33]

Once a basis of V is chosen, linear maps $f : V \to W$ are completely determined by specifying the images of the basis vectors, because any element of V is expressed uniquely as a linear combination of them.[34] If dim V = dim W, a 1-to-1 correspondence between fixed bases of V and W gives rise to a linear map that maps any basis element of V to the corresponding basis element of W. It is an isomorphism, by its very definition.[35] Therefore, two vector spaces are isomorphic if their dimensions agree and vice versa. Another way to express this is that any vector space is *completely classified* (up to isomorphism) by its dimension, a single number. In particular, any n-dimensional F-vector space V is isomorphic to F^n. There is, however, no "canonical" or preferred isomorphism; actually an isomorphism $\varphi: F^n \to V$ is equivalent to the choice of a basis of V, by mapping the standard basis of F^n to V, via φ. Appending an automorphism, i.e. an isomorphism $\psi: V \to V$ yields another isomorphism $\psi \circ \varphi: F^n \to V$, the composition of ψ and φ, and therefore a different basis of V. The freedom of choosing a convenient basis is particularly useful in the infinite-dimensional context, see below.

Matrices

Matrices are a useful notion to encode linear maps.[36] They are written as a rectangular array of scalars as in the image at the right. Any m-by-n matrix A gives rise to a linear map from F^n to F^m, by the following



A typical matrix

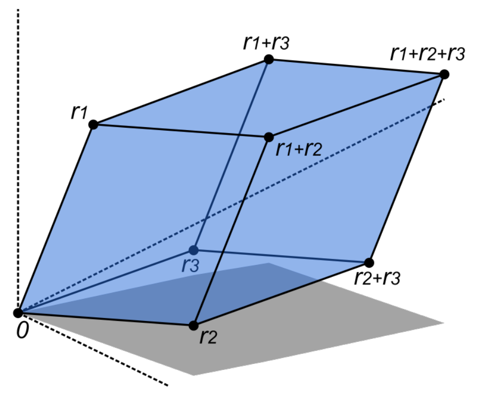
$$\mathbf{x} = (x_1, x_2, \cdots, x_n) \mapsto \left(\sum_{j=1}^{n} a_{1j} x_j, \sum_{j=1}^{n} a_{2j} x_j, \cdots, \sum_{j=1}^{n} a_{mj} x_j \right), \text{ where } \sum \text{denotes summation,}$$

or, using the matrix multiplication of the matrix A with the coordinate vector $\mathbf{x}$:

$$\mathbf{x} \mapsto A\mathbf{x}.$$

Moreover, after choosing bases of V and W, *any* linear map $f : V \to W$ is uniquely represented by a matrix via this assignment.[37]

The determinant $\det(A)$ of a square matrix A is a scalar that tells whether the associated map is an isomorphism or not: to be so it is sufficient and necessary that the determinant is nonzero.[38] The linear transformation of $\mathbf{R}^n$ corresponding to a real n-by-n matrix is orientation preserving if and only if the determinant is positive.



The volume of this parallelepiped is the absolute value of the determinant of the 3-by-3 matrix formed by the vectors r_1, r_2, and r_3.

Eigenvalues and eigenvectors

Endomorphisms, linear maps $f : V \to V$, are particularly important since in this case vectors $\mathbf{v}$ can be compared with their image under f, $f(\mathbf{v})$. Any nonzero vector $\mathbf{v}$ satisfying $\lambda\mathbf{v} = f(\mathbf{v})$, where λ is a scalar, is called an *eigenvector* of f with *eigenvalue* λ.[39] [40] Equivalently, $\mathbf{v}$ is an element of the kernel of the difference $f - \lambda \cdot \mathrm{Id}$ (where Id is the identity map $V \to V$). If V is finite-dimensional, this can be rephrased using determinants: f having eigenvalue λ is equivalent to

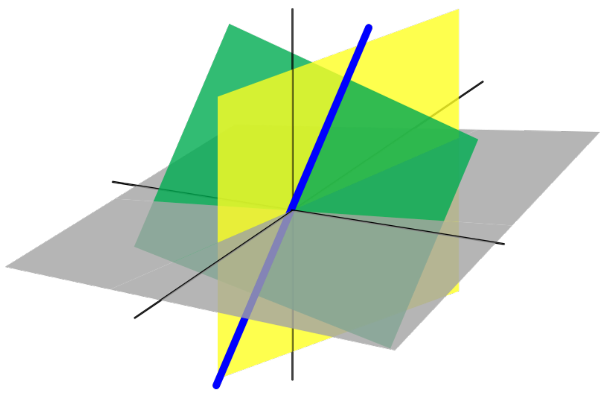
$$\det(f - \lambda \cdot \mathrm{Id}) = 0.$$

By spelling out the definition of the determinant, the expression on the left hand side can be seen to be a polynomial function in λ, called the characteristic polynomial of f.[41] If the field F is large enough to contain a zero of this polynomial (which automatically happens for F algebraically closed, such as $F = \mathbf{C}$) any linear map has at least one eigenvector. The vector space V may or may not possess an eigenbasis, a basis consisting of eigenvectors. This phenomenon is governed by the Jordan canonical form of the map.[42] The set of all eigenvectors corresponding to a particular eigenvalue of f forms a vector space known as the *eigenspace* corresponding to the eigenvalue (and f) in question. To achieve the spectral theorem, the corresponding statement in the infinite-dimensional case, the machinery of functional analysis is needed, see below.

Basic constructions

In addition to the above concrete examples, there are a number of standard linear algebraic constructions that yield vector spaces related to given ones. In addition to the definitions given below, they are also characterized by universal properties, which determine an object X by specifying the linear maps from X to any other vector space.

Subspaces and quotient spaces



A line passing through the origin (blue, thick) in $\mathbf{R}^3$ is a linear subspace. It is the intersection of two planes (green and yellow).

A nonempty subset W of a vector space V that is closed under addition and scalar multiplication (and therefore contains the **0**-vector of V) is called a *subspace* of V.[43] Subspaces of V are vector spaces (over the same field) in their own right. The intersection of all subspaces containing a given set S of vectors is called its span, and is the smallest subspace of V containing the set S. Expressed in terms of elements, the span is the subspace consisting of all the linear combinations of elements of S.[44]

The counterpart to subspaces are *quotient vector spaces*.[45] Given any subspace $W \subset V$, the quotient space V/W ("V modulo W") is defined as follows: as a set, it consists of $\mathbf{v} + W = \{\mathbf{v} + \mathbf{w}, \mathbf{w} \in W\}$, where $\mathbf{v}$ is an arbitrary vector in V. The sum of two such elements $\mathbf{v}_1 + W$ and $\mathbf{v}_2 + W$ is $(\mathbf{v}_1 + \mathbf{v}_2) + W$, and scalar multiplication is given by $a \cdot (\mathbf{v} + W) = (a \cdot \mathbf{v}) + W$. The key point in this definition is that $\mathbf{v}_1 + W = \mathbf{v}_2 + W$ if and only if the difference of $\mathbf{v}_1$ and $\mathbf{v}_2$ lies in W.[46] This way, the quotient space "forgets" information that is contained in the subspace W.

The kernel ker(f) of a linear map f: $V \to W$ consists of vectors $\mathbf{v}$ that are mapped to **0** in W.[47] Both kernel and image im(f) = $\{f(\mathbf{v}), \mathbf{v} \in V\}$ are subspaces of V and W, respectively.[48] The existence of kernels and images is part of the statement that the category of vector spaces (over a fixed field F) is an abelian category, i.e. a corpus of mathematical objects and structure-preserving maps between them (a category) that behaves much like the category of abelian groups.[49] Because of this, many statements such as the first isomorphism theorem (also called rank-nullity theorem in matrix-related terms)

$$V / \ker(f) \cong \operatorname{im}(f).$$

and the second and third isomorphism theorem can be formulated and proven in a way very similar to the corresponding statements for groups.

An important example is the kernel of a linear map $\mathbf{x} \mapsto A\mathbf{x}$ for some fixed matrix A, as above. The kernel of this map is the subspace of vectors $\mathbf{x}$ such that $A\mathbf{x} = 0$, which is precisely the set of solutions to the system of homogeneous linear equations belonging to A. This concept also extends to linear differential equations

$$a_0 f + a_1 \frac{df}{dx} + a_2 \frac{d^2 f}{dx^2} + \cdots + a_n \frac{d^n f}{dx^n} = 0,$$ where the coefficients a_i are functions in x, too.

In the corresponding map

$$f \mapsto D(f) = \sum_{i=0}^{n} a_i \frac{d^i f}{dx^i},$$

the derivatives of the function f appear linearly (as opposed to $f''(x)^2$, for example). Since differentiation is a linear procedure (i.e., $(f + g)' = f' + g'$ and $(c \cdot f)' = c \cdot f'$ for a constant c) this assignment is linear, called a linear differential operator. In particular, the solutions to the differential equation $D(f) = 0$ form a vector space (over **R** or **C**).

Direct product and direct sum

The *direct product* $\prod_{i\in I} V_i$ of a family of vector spaces V_i consists of the set of all tuples $(\mathbf{v}_i)_{i\in I}$, which specify for each index i in some index set I an element $\mathbf{v}_i$ of V_i.[50] Addition and scalar multiplication is performed componentwise. A variant of this construction is the *direct sum* $\bigoplus_{i\in I} V_i$ (also called coproduct and denoted $\coprod_{i\in I} V_i$), where only tuples with finitely many nonzero vectors are allowed. If the index set I is finite, the two constructions agree, but differ otherwise.

Tensor product

The *tensor product* $V \otimes_F W$, or simply $V \otimes W$, of two vector spaces V and W is one of the central notions of multilinear algebra which deals with extending notions such as linear maps to several variables. A map g: $V \times W \to X$ is called bilinear if g is linear in both variables $\mathbf{v}$ and $\mathbf{w}$. That is to say, for fixed $\mathbf{w}$ the map $\mathbf{v} \mapsto g(\mathbf{v}, \mathbf{w})$ is linear in the sense above and likewise for fixed $\mathbf{v}$.

The tensor product is a particular vector space that is a *universal* recipient of bilinear maps g, as follows. It is defined as the vector space consisting of finite (formal) sums of symbols called tensors

$$\mathbf{v}_1 \otimes \mathbf{w}_1 + \mathbf{v}_2 \otimes \mathbf{w}_2 + \ldots + \mathbf{v}_n \otimes \mathbf{w}_n,$$

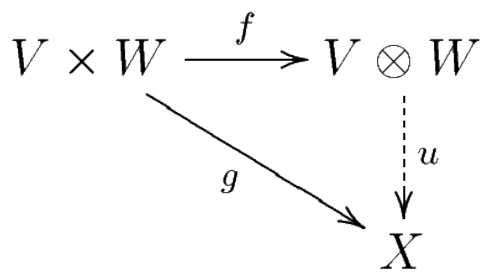
subject to the rules

$$a \cdot (\mathbf{v} \otimes \mathbf{w}) = (a \cdot \mathbf{v}) \otimes \mathbf{w} = \mathbf{v} \otimes (a \cdot \mathbf{w}), \text{ where } a \text{ is a scalar,}$$

$$(\mathbf{v}_1 + \mathbf{v}_2) \otimes \mathbf{w} = \mathbf{v}_1 \otimes \mathbf{w} + \mathbf{v}_2 \otimes \mathbf{w}, \text{ and}$$

$$\mathbf{v} \otimes (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v} \otimes \mathbf{w}_1 + \mathbf{v} \otimes \mathbf{w}_2.$$ [51]

These rules ensure that the map f from the $V \times W$ to $V \otimes W$ that maps a tuple $(\mathbf{v}, \mathbf{w})$ to $\mathbf{v} \otimes \mathbf{w}$ is bilinear. The universality states that given *any* vector space X and *any* bilinear map g: $V \times W \to X$, there exists a unique map u, shown in the diagram with a dotted arrow, whose composition with f equals g: $u(\mathbf{v} \otimes \mathbf{w}) = g(\mathbf{v}, \mathbf{w})$.[52] This is called the universal property of the tensor product, an instance of the method—much used in advanced abstract algebra—to indirectly define objects by specifying maps from or to this object.



Commutative diagram depicting the universal property of the tensor product.

Vector spaces with additional structure

From the point of view of linear algebra, vector spaces are completely understood insofar as any vector space is characterized, up to isomorphism, by its dimension. However, vector spaces *ad hoc* do not offer a framework to deal with the question—crucial to analysis—whether a sequence of functions converges to another function. Likewise, linear algebra is not adapted to deal with infinite series, since the addition operation allows only finitely many terms to be added. Therefore, the needs of functional analysis require considering additional structures. Much the same way the axiomatic treatment of vector spaces reveals their essential algebraic features, studying vector spaces with additional data abstractly turns out to be advantageous, too.

A first example of an additional datum is an order ≤, a token by which vectors can be compared.[53] For example, n-dimensional real space $\mathbf{R}^n$ can be ordered by comparing its vectors componentwise. Ordered vector spaces, for example Riesz spaces, are fundamental to Lebesgue integration, which relies on the ability to express a function as a difference of two positive functions

$$f = f^+ - f^-,$$

where f^+ denotes the positive part of f and f^- the negative part.[54]

Normed vector spaces and inner product spaces

"Measuring" vectors is done by specifying a norm, a datum which measures lengths of vectors, or by an inner product, which measures angles between vectors. Norms and inner products are denoted $|\mathbf{v}|$ and $\langle \mathbf{v}|\mathbf{w}\rangle$, respectively. The datum of an inner product entails that lengths of vectors can be defined too, by defining the associated norm $|\mathbf{v}| := \sqrt{\langle \mathbf{v}|\mathbf{v}\rangle}$. Vector spaces endowed with such data are known as *normed vector spaces* and *inner product spaces*, respectively.[55]

Coordinate space F^n can be equipped with the standard dot product:

$$\langle \mathbf{x}|\mathbf{y}\rangle = \mathbf{x}\cdot\mathbf{y} = x_1y_1 + \cdots + x_ny_n.$$

In $\mathbf{R}^2$, this reflects the common notion of the angle between two vectors **x** and **y**, by the law of cosines:

$$\mathbf{x}\cdot\mathbf{y} = \cos\left(\angle(\mathbf{x},\mathbf{y})\right)\cdot|\mathbf{x}|\cdot|\mathbf{y}|.$$

Because of this, two vectors satisfying $\langle \mathbf{x}|\mathbf{y}\rangle = 0$ are called orthogonal. An important variant of the standard dot product is used in Minkowski space: $\mathbf{R}^4$ endowed with the Lorentz product

$$\langle \mathbf{x}|\mathbf{y}\rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4.$$ [56]

In contrast to the standard dot product, it is not positive definite: $\langle \mathbf{x}|\mathbf{x}\rangle$ also takes negative values, for example for **x** = (0, 0, 0, 1). Singling out the fourth coordinate—corresponding to time, as opposed to three space-dimensions—makes it useful for the mathematical treatment of special relativity.

Topological vector spaces

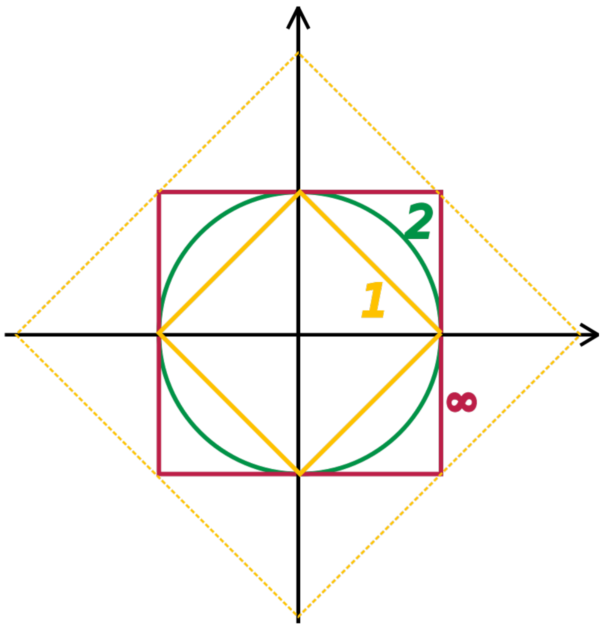
Convergence questions are treated by considering vector spaces *V* carrying a compatible topology, a structure that allows one to talk about elements being close to each other.[57] [58] Compatible here means that addition and scalar multiplication have to be continuous maps. Roughly, if **x** and **y** in *V*, and *a* in *F* vary by a bounded amount, then so do **x** + **y** and *a***x**.[59] To make sense of specifying the amount a scalar changes, the field *F* also has to carry a topology in this context; a common choice are the reals or the complex numbers.

In such *topological vector spaces* one can consider series of vectors. The infinite sum

$$\sum_{i=0}^{\infty} f_i$$

denotes the limit of the corresponding finite partial sums of the sequence $(f_i)_{i\in\mathbf{N}}$ of elements of *V*. For example, the f_i could be (real or complex) functions belonging to some function space *V*, in which case the series is a function series. The mode of convergence of the series depends on the topology imposed on the function space. In such cases, pointwise convergence and uniform convergence are two prominent examples.

A way to ensure the existence of limits of certain infinite series is to restrict attention to spaces where any Cauchy sequence has a limit; such a vector space is called complete. Roughly, a vector space is complete provided that it contains all necessary limits. For example, the vector space of polynomials on the unit interval [0,1], equipped with the topology of uniform convergence is not complete because any continuous function on [0,1] can be uniformly approximated by a sequence of polynomials, by the Weierstrass approximation theorem.[60] In contrast, the space of *all* continuous functions on [0,1] with the same topology is complete.[61] A norm gives rise to a topology by defining that a sequence of vectors $\mathbf{v}_n$ converges to $\mathbf{v}$ if and only if



Unit "spheres" in $\mathbf{R}^2$ consist of plane vectors of norm 1. Depicted are the unit spheres in different p-norms, for p = 1, 2, and ∞. The bigger diamond depicts points of 1-norm equal to $\sqrt{2}$.

$$\lim_{n\to\infty}|\mathbf{v}_n - \mathbf{v}| = 0.$$

Banach and Hilbert spaces are complete topological spaces whose topologies are given, respectively, by a norm and an inner product. Their study—a key piece of functional analysis—focusses on infinite-dimensional vector spaces, since all norms on finite-dimensional topological vector spaces give rise to the same notion of convergence.[62] The image at the right shows the equivalence of the 1-norm and ∞-norm on $\mathbf{R}^2$: as the unit "balls" enclose each other, a sequence converges to zero in one norm if and only if it so does in the other norm. In the infinite-dimensional case, however, there will generally be inequivalent topologies, which makes the study of topological vector spaces richer than that of vector spaces without additional data.

From a conceptual point of view, all notions related to topological vector spaces should match the topology. For example, instead of considering all linear maps (also called functionals) $V \to W$, maps between topological vector spaces are required to be continuous.[63] In particular, the (topological) dual space V^* consists of continuous functionals $V \to \mathbf{R}$ (or $\mathbf{C}$). The fundamental Hahn-Banach theorem is concerned with separating subspaces of appropriate topological vector spaces by continuous functionals.[64]

Banach spaces

Banach spaces, introduced by Stefan Banach, are complete normed vector spaces.[65] A first example is the vector space ℓ^p consisting of infinite vectors with real entries $\mathbf{x} = (x_1, x_2, ...)$ whose p-norm ($1 \le p \le \infty$) given by

$$|\mathbf{x}|_p := \left(\sum_i |x_i|^p\right)^{1/p} \text{ for } p < \infty \text{ and } |\mathbf{x}|_\infty := \sup_i |x_i|$$

is finite. The topologies on the infinite-dimensional space ℓ^p are inequivalent for different p. E.g. the sequence of vectors $\mathbf{x}_n = (2^{-n}, 2^{-n}, ..., 2^{-n}, 0, 0, ...)$, i.e. the first 2^n components are 2^{-n}, the following ones are 0, converges to the zero vector for $p = \infty$, but does not for $p = 1$:

$$|x_n|_\infty = \sup(2^{-n}, 0) = 2^{-n} \to 0, \text{ but } |x_n|_1 = \sum_{i=1}^{2^n} 2^{-n} = 2^n \cdot 2^{-n} = 1.$$

More generally than sequences of real numbers, functions $f: \Omega \to \mathbf{R}$ are endowed with a norm that replaces the above sum by the Lebesgue integral

$$|f|_p := \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}.$$

The space of integrable functions on a given domain Ω (for example an interval) satisfying $|f|_p < \infty$, and equipped with this norm are called Lebesgue spaces, denoted $L^p(\Omega)$.[66] These spaces are complete.[67] (If one uses the Riemann integral instead, the space is *not* complete, which may be seen as a justification for Lebesgue's integration theory.[68]) Concretely this means that for any sequence of Lebesgue-integrable functions $f_1, f_2, \ldots$ with $|f_n|_p < \infty$, satisfying the condition

$$\lim_{k,\ n \to \infty} \int_\Omega |f_k(x) - f_n(x)|^p \, dx = 0$$

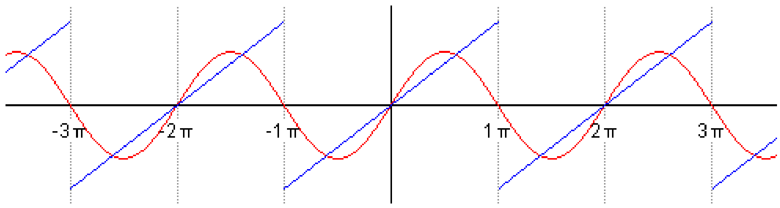
there exists a function $f(x)$ belonging to the vector space $L^p(\Omega)$ such that

$$\lim_{k \to \infty} \int_\Omega |f(x) - f_k(x)|^p \, dx = 0.$$

Imposing boundedness conditions not only on the function, but also on its derivatives leads to Sobolev spaces.[69]

Hilbert spaces

Complete inner product spaces are known as *Hilbert spaces*, in honor of David Hilbert.[70] The Hilbert space $L^2(\Omega)$, with inner product given by



The succeeding snapshots show summation of 1 to 5 terms in approximating a periodic function (blue) by finite sum of sine functions (red).

$$\langle f\ ,\ g \rangle = \int_\Omega f(x) \overline{g(x)} \, dx,$$

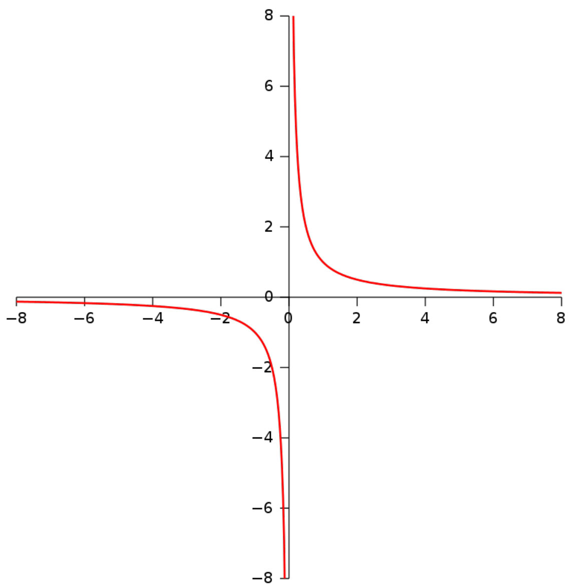
where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$.[71] [72] is a key case.

By definition, in a Hilbert space any Cauchy sequences converges to a limit. Conversely, finding a sequence of functions f_n with desirable properties that approximates a given limit function, is equally crucial. Early analysis, in the guise of the Taylor approximation, established an approximation of differentiable functions f by polynomials.[73] By the Stone–Weierstrass theorem, every continuous function on $[a, b]$ can be approximated as closely as desired by a polynomial.[74] A similar approximation technique by trigonometric functions is commonly called Fourier expansion, and is much applied in engineering, see below. More generally, and more conceptually, the theorem yields a simple description of what "basic functions", or, in abstract Hilbert spaces, what basic vectors suffice to generate a Hilbert space H, in the sense that the *closure* of their span (i.e., finite linear combinations and limits of those) is the whole space. Such a set of functions is called a *basis* of H, its cardinality is known as the Hilbert dimension.[75] Not only does the theorem exhibit suitable basis functions as sufficient for approximation purposes, together with the Gram-Schmidt process it also allows to construct a basis of orthogonal vectors.[76] Such orthogonal bases are the Hilbert space generalization of the coordinate axes in finite-dimensional Euclidean space.

The solutions to various differential equations can be interpreted in terms of Hilbert spaces. For example, a great many fields in physics and engineering lead to such equations and frequently solutions with particular physical properties are used as basis functions, often orthogonal.[77] As an example from physics, the time-dependent Schrödinger equation in quantum mechanics describes the change of physical properties in time, by means of a partial differential equation whose solutions are called wavefunctions.[78] Definite values for physical properties such as energy, or momentum, correspond to eigenvalues of a certain (linear) differential operator and the associated wavefunctions are called eigenstates. The spectral theorem decomposes a linear compact operator acting on functions in terms of these eigenfunctions and their eigenvalues.[79]

Algebras over fields

General vector spaces do not possess a multiplication operation. A vector space equipped with an additional bilinear operator defining the multiplication of two vectors is an *algebra over a field*.[80] Many algebras stem from functions on some geometrical object: since functions with values in a field can be multiplied, these entities form algebras. The Stone–Weierstrass theorem mentioned above, for example, relies on Banach algebras which are both Banach spaces and algebras.



A hyperbola, given by the equation $x \cdot y = 1$. The coordinate ring of functions on this hyperbola is given by $\mathbf{R}[x, y] / (x \cdot y - 1)$, an infinite-dimensional vector space over $\mathbf{R}$.

Commutative algebra makes great use of rings of polynomials in one or several variables, introduced above. Their multiplication is both commutative and associative. These rings and their quotients form the basis of algebraic geometry, because they are rings of functions of algebraic geometric objects.[81]

Another crucial example are *Lie algebras*, which are neither commutative nor associative, but the failure to be so is limited by the constraints ($[x, y]$ denotes the product of x and y):

- $[x, y] = -[y, x]$ (anticommutativity) and
- $[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0$ (Jacobi identity).[82]

Examples include the vector space of n-by-n matrices, with $[x, y] = xy - yx$, the commutator of two matrices, and $\mathbf{R}^3$, endowed with the cross product.

The tensor algebra $T(V)$ is a formal way of adding products to any vector space V to obtain an algebra.[83] As a vector space, it is spanned by symbols, called simple tensors

$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \ldots \otimes \mathbf{v}_n$, where the degree n varies.

The multiplication is given by concatenating such symbols, imposing the distributive law under addition, and requiring that scalar multiplication commute with the tensor product ⊗, much the same way as with the tensor product of two vector spaces introduced above. In general, there are no relations between $\mathbf{v}_1 \otimes \mathbf{v}_2$ and $\mathbf{v}_2 \otimes \mathbf{v}_1$. Forcing two such elements to be equal leads to the symmetric algebra, whereas forcing $\mathbf{v}_1 \otimes \mathbf{v}_2 = -\mathbf{v}_2 \otimes \mathbf{v}_1$ yields the exterior algebra.[84]

Applications

Vector spaces have manifold applications as they occur in many circumstances, namely wherever functions with values in some field are involved. They provide a framework to deal with analytical and geometrical problems, or are used in the Fourier transform. This list is not exhaustive: many more applications exist, for example in optimization. The minimax theorem of game theory stating the existence of a unique payoff when all players play optimally can be formulated and proven using vector spaces methods.[85] Representation theory fruitfully transfers the good understanding of linear algebra and vector spaces to other mathematical domains such as group theory.[86]

Distributions

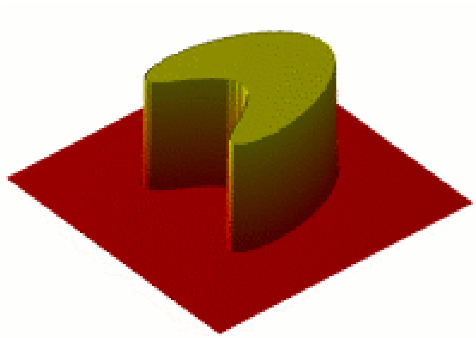
A *distribution* (or *generalized function*) is a linear map assigning a number to each "test" function, typically a smooth function with compact support, in a continuous way: in the above terminology the space of distributions is the (continuous) dual of the test function space.[87] The latter space is endowed with a topology that takes into account not only *f* itself, but also all its higher derivatives. A standard example is the result of integrating a test function *f* over some domain Ω:

$$I(f) = \int_\Omega f(x)\,dx.$$

When Ω = {*p*}, the set consisting of a single point, this reduces to the Dirac distribution, denoted by δ, which associates to a test function *f* its value at the *p*: δ(*f*) = *f*(*p*). Distributions are a powerful instrument to solve differential equations. Since all standard analytic notions such as derivatives are linear, they extend naturally to the space of distributions. Therefore the equation in question can be transferred to a distribution space, which is bigger than the underlying function space, so that more flexible methods are available for solving the equation. For example, Green's functions and fundamental solutions are usually distributions rather than proper functions, and can then be used to find solutions of the equation with prescribed boundary conditions. The found solution can then in some cases be proven to be actually a true function, and a solution to the original equation (e.g., using the Lax-Milgram theorem, a consequence of the Riesz representation theorem).[88]

Fourier analysis

Resolving a periodic function into a sum of trigonometric functions forms a *Fourier series*, a technique much used in physics and engineering.[89] [90] The underlying vector space is usually the Hilbert space $L^2(0, 2\pi)$, for which the functions sin *mx* and cos *mx* (*m* an integer) form an orthogonal basis.[91] The Fourier expansion of an L^2 function *f* is



The heat equation describes the dissipation of physical properties over time, such as the decline of the temperature of a hot body placed in a colder environment (yellow depicts hotter regions than red).

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m \cos(mx) + b_m \sin(mx)\right].$$

The coefficients a_m and b_m are called Fourier coefficients of *f*, and are calculated by the formulas[92]

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(mt)\,dt\ ,\ b_m = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(mt)\,dt.$$

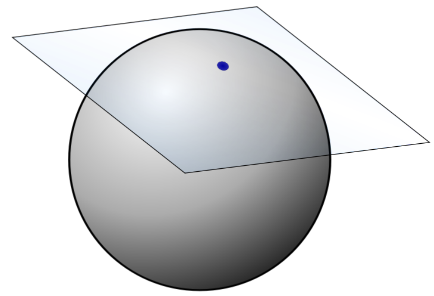
In physical terms the function is represented as a superposition of sine waves and the coefficients give information about the function's frequency spectrum.[93] A complex-number form of Fourier series is also commonly used.[94] The concrete formulae above are consequences of a more general mathematical duality called Pontryagin duality.[95] Applied to the group **R**, it yields the classical Fourier transform; an application in physics are reciprocal lattices, where the underlying group is a finite-dimensional real vector space endowed with the additional datum of a lattice encoding positions of atoms in crystals.[96]

Fourier series are used to solve boundary value problems in partial differential equations.[97] In 1822, Fourier first used this technique to solve the heat equation.[98] A discrete version of the Fourier series can be used in sampling applications where the function value is known only at a finite number of equally spaced points. In this case the Fourier series is finite and its value is equal to the sampled values at all points.[99] The set of coefficients is known as the discrete Fourier transform (DFT) of the given sample sequence. The DFT is one of the key tools of digital signal processing, a field whose applications include radar, speech encoding, image compression.[100] The JPEG image format is an application of the closely-related discrete cosine transform.[101]

The fast Fourier transform is an algorithm for rapidly computing the discrete Fourier transform.[102] It is used not only for calculating the Fourier coefficients but, using the convolution theorem, also for computing the convolution of two finite sequences.[103] They in turn are applied in digital filters[104] and as a rapid multiplication algorithm for polynomials and large integers (Schönhage-Strassen algorithm).[105] [106]

Differential geometry

The tangent plane to a surface at a point is naturally a vector space whose origin is identified with the point of contact. The tangent plane is the best linear approximation, or linearization, of a surface at a point.[107] Even in a three-dimensional Euclidean space, there is typically no natural way to prescribe a basis of the tangent plane, and so it is conceived of as an abstract vector space rather than a real coordinate space. The *tangent space* is the generalization to higher-dimensional differentiable manifolds.[108]



The tangent space to the 2-sphere at some point is the infinite plane touching the sphere in this point.

Riemannian manifolds are manifolds whose tangent spaces are endowed with a suitable inner product.[109] Derived therefrom, the Riemann curvature tensor encodes all curvatures of a manifold in one object, which finds applications in general relativity, for example, where the Einstein curvature tensor describes the matter and energy content of space-time.[110] [111] The tangent space of a Lie group can be given naturally the structure of a Lie algebra and can be used to classify compact Lie groups.[112]

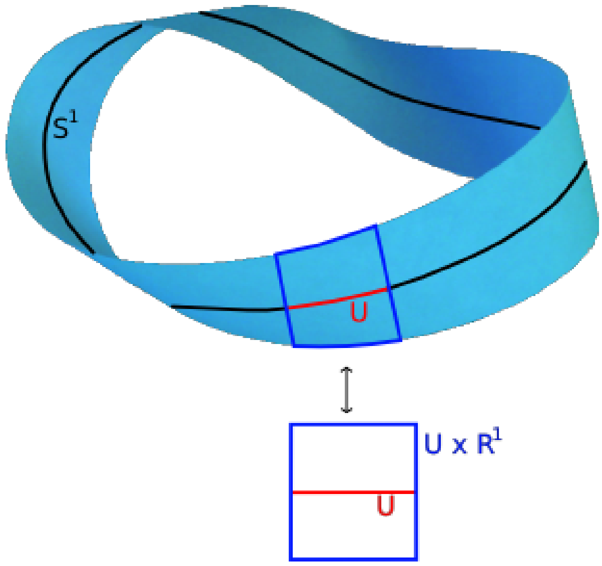
Generalizations

Vector bundles

A *vector bundle* is a family of vector spaces parametrized continuously by a topological space X.[113] More precisely, a vector bundle over X is a topological space E equipped with a continuous map

$$\pi : E \to X$$

such that for every x in X, the fiber $\pi^{-1}(x)$ is a vector space. The case dim $V = 1$ is called a line bundle. For any vector space V, the projection $X \times V \to X$ makes the product $X \times V$ into a "trivial" vector bundle. Vector bundles over X are required to be locally a product of X and some (fixed) vector space V: for every x in X, there is a neighborhood U of x such that the restriction of π to $\pi^{-1}(U)$ is isomorphic[114] to the trivial bundle $U \times V \to U$. Despite their locally trivial character, vector bundles may (depending on the shape of the underlying space X) be "twisted" in the large, i.e., the bundle need not be (globally isomorphic to) the trivial bundle $X \times V$. For example, the Möbius strip can be seen as a line bundle over the circle S^1 (by identifying open intervals with the real line). It is, however, different from the cylinder $S^1 \times \mathbf{R}$, because the latter is orientable whereas the former is not.[115]



A Möbius strip. Locally, it looks like $U \times \mathbf{R}$.

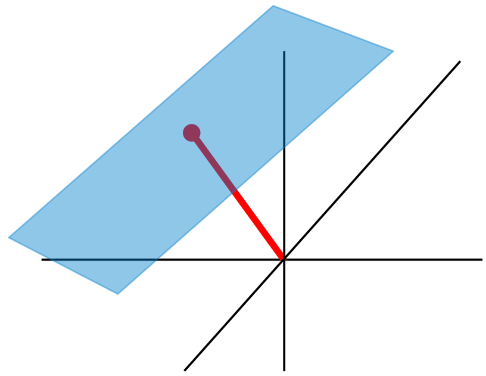
Properties of certain vector bundles provide information about the underlying topological space. For example, the tangent bundle consists of the collection of tangent spaces parametrized by the points of a differentiable manifold. The tangent bundle of the circle S^1 is globally isomorphic to $S^1 \times \mathbf{R}$, since there is a global nonzero vector field on S^1.[116] In contrast, by the hairy ball theorem, there is no (tangent) vector field on the 2-sphere S^2 which is everywhere nonzero.[117] K-theory studies the isomorphism classes of all vector bundles over some topological space.[118] In addition to deepening topological and geometrical insight, it has purely algebraic consequences, such as the classification of finite-dimensional real division algebras: **R**, **C**, the quaternions **H** and the octonions.

The cotangent bundle of a differentiable manifold consists, at every point of the manifold, of the dual of the tangent space, the cotangent space. Sections of that bundle are known as differential forms. They are used to do integration on manifolds.

Modules

Modules are to rings what vector spaces are to fields. The very same axioms, applied to a ring R instead of a field F yield modules.[119] The theory of modules, compared to vector spaces, is complicated by the presence of ring elements that do not have multiplicative inverses. For example, modules need not have bases, as the **Z**-module (i.e., abelian group) **Z**/2**Z** shows; those modules that do (including all vector spaces) are known as free modules. Nevertheless, a vector space can be compactly defined as a module over a ring which is a field with the elements being called vectors. The algebro-geometric interpretation of commutative rings via their spectrum allows the development of concepts such as locally free modules, the algebraic counterpart to vector bundles.

Affine and projective spaces



An affine plane (light blue) in $\mathbf{R}^3$. It is a two-dimensional subspace shifted by a vector **x** (red).

Roughly, *affine spaces* are vector spaces whose origin is not specified.[120] More precisely, an affine space is a set with a free transitive vector space action. In particular, a vector space is an affine space over itself, by the map

$$V \times V \to V, (\mathbf{v}, \mathbf{a}) \mapsto \mathbf{a} + \mathbf{v}.$$

If W is a vector space, then an affine subspace is a subset of W obtained by translating a linear subspace V by a fixed vector $\mathbf{x} \in W$; this space is denoted by $\mathbf{x} + V$ (it is a coset of V in W) and consists of all vectors of the form $\mathbf{x} + \mathbf{v}$ for $\mathbf{v} \in V$. An important example is the space of solutions of a system of inhomogeneous linear equations

$$A\mathbf{x} = \mathbf{b}$$

generalizing the homogeneous case $\mathbf{b} = 0$ above.[121] The space of solutions is the affine subspace $\mathbf{x} + V$ where $\mathbf{x}$ is a particular solution of the equation, and V is the space of solutions of the homogeneous equation (the nullspace of A).

The set of one-dimensional subspaces of a fixed finite-dimensional vector space V is known as *projective space*; it may be used to formalize the idea of parallel lines intersecting at infinity.[122] Grassmannians and flag manifolds generalize this by parametrizing linear subspaces of fixed dimension k and flags of subspaces, respectively.

See also

- Coordinates (mathematics)
- Euclidean vector, for vectors in physics
- Gyrovector space
- Graded vector space
- Metric space
- P-vector
- Riesz–Fischer theorem
- Vector spaces without fields
- Space (mathematics)

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[1] Some authors (such as Brown 1991) restrict attention to the fields **R** or **C**, but most of the theory is unchanged over an arbitrary field.

[2] It is also common, especially in physics, to denote vectors with an arrow on top: $\vec{v}$.

[3] Roman 2005, ch. 1, p. 27.

[4] This axiom is not asserting the associativity of an operation, since there are two operations in question, scalar multiplication: $b\mathbf{v}$; and field multiplication: ab.

[5] van der Waerden 1993, Ch. 19.

[6] Bourbaki 1998, Section II.1.1. Bourbaki calls the group homomorphisms $f(a)$ *homotheties*.

[7] Bourbaki 1969, ch. "Algèbre linéaire et algèbre multilinéaire", pp. 78–91.

[8] Bolzano 1804.

[9] Möbius 1827.

[10] Hamilton 1853.

[11] Grassmann 2000.

[12] Peano 1888, ch. IX.

[13] Banach 1922.

[14] Dorier 1995, Moore 1995.

[15] Lang 1987, ch. I.1

[16] e.g. Lang 1993, ch. XII.3., p. 335

[17] Lang 1987, ch. IX.1

[18] Lang 1987, ch. VI.3.

[19] Lang 2002, ch. V.1

[20] Lang 1987, ch. II.2., pp. 47–48

[21] Roman 2005, Theorem 1.9, p. 43.

[22] Blass 1984.

[23] Halpern 1966, pp. 670–673.

[24] Artin 1991, Theorem 3.3.13.

[25] The indicator functions of intervals (of which there are infinitely many) are linearly independent, for example.

[26] Braun 1993, Th. 3.4.5, p. 291.

[27] Stewart 1975, Proposition 4.3, p. 52.

[28] Stewart 1975, Theorem 6.5, p. 74.

[29] Roman 2005, ch. 2, p. 45.

[30] Lang 1987, ch. IV.4, Corollary, p. 106.

[31] Lang 1987, Example IV.2.6.

[32] Lang 1987, ch. VI.6.

[33] Halmos 1974, p. 28, Ex. 9.

[34] Lang 1987, Theorem IV.2.1, p. 95.

[35] Roman 2005, Th. 2.5 and 2.6, p. 49.

[36] Lang 1987, ch. V.1.

[37] Lang 1987, ch. V.3., Corollary, p. 106.
[38] Lang 1987, Theorem VII.9.8, p. 198.
[39] The nomenclature derives from German "eigen", which means own or proper.
[40] Roman 2005, ch. 8, p. 135–156.
[41] Lang 1987, ch. IX.4.
[42] Roman 2005, ch. 8, p. 140. See also Jordan–Chevalley decomposition.
[43] Roman 2005, ch. 1, p. 29.
[44] Roman 2005, ch. 1, p. 35.
[45] Roman 2005, ch. 3, p. 64.
[46] Some authors (such as Roman 2005) choose to start with this equivalence relation and derive the concrete shape of V/W from this.
[47] Lang 1987, ch. IV.3..
[48] Roman 2005, ch. 2, p. 48.
[49] Mac Lane 1998.
[50] Roman 2005, ch. 1, pp. 31–32.
[51] Lang 2002, ch. XVI.1
[52] Roman 2005, Th. 14.3. See also Yoneda lemma.
[53] Schaefer & Wolff 1999, pp. 204–205
[54] Bourbaki 2004, ch. 2, p. 48.
[55] Roman 2005, ch. 9.
[56] Naber 2003, ch. 1.2.
[57] Treves 1967.
[58] Bourbaki 1987.
[59] This requirement implies that the topology gives rise to a uniform structure, Bourbaki 1989, ch. II.
[60] Kreyszig 1989, §4.11-5
[61] Kreyszig 1989, §1.5-5
[62] Choquet 1966, Proposition III.7.2.
[63] Treves 1967, p. 34–36.
[64] Lang 1983, Cor. 4.1.2, p. 69.
[65] Treves 1967, ch. 11.
[66] The triangle inequality for $|-|_p$ is provided by the Minkowski inequality. For technical reasons, in the context of functions one has to identify functions that agree almost everywhere to get a norm, and not only a seminorm.
[67] Treves 1967, Theorem 11.2, p. 102
[68] "Many functions in L^2 of Lebesgue measure, being unbounded, cannot be integrated with the classical Riemann integral. So spaces of Riemann integrable functions would not be complete in the L^2 norm, and the orthogonal decomposition would not apply to them. This shows one of the advantages of Lebesgue integration.", Dudley 1989, sect. 5.3, p. 125.
[69] Evans 1998, ch. 5
[70] Treves 1967, ch. 12.
[71] Dennery 1996, p.190.
[72] For $p \neq 2$, $L^p(\Omega)$ is not a Hilbert space.
[73] Lang 1993, Th. XIII.6, p. 349
[74] Lang 1993, Th. III.1.1
[75] A basis of a Hilbert space is not the same thing as a basis in the sense of linear algebra above. For distinction, the latter is then called a Hamel basis.
[76] Choquet 1966, Lemma III.16.11
[77] Kreyszig 1999, Chapter 11.
[78] Griffiths 1995, Chapter 1.
[79] Lang 1993, ch. XVII.3
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[83] Lang 2002, ch. XVI.7.
[84] Lang 2002, ch. XVI.8.
[85] Luenberger 1997, Section 7.13.
[86] See representation theory and group representation.
[87] Lang 1993, Ch. XI.1
[88] Evans 1998, Th. 6.2.1
[89] Although the Fourier series is periodic, the technique can be applied to any L^2 function on an interval by considering the function to be continued periodically outside the interval. See Kreyszig 1988, p. 601
[90] Folland 1992, p. 349 *ff*.

[91] Gasquet & Witomski 1999, p. 150
[92] Gasquet & Witomski 1999, §4.5
[93] Gasquet & Witomski 1999, p. 57
[94] Gasquet & Witomski 1999, §4.5
[95] Loomis 1953, Ch. VII.
[96] Ashcroft & Mermin 1976, Ch. 5.
[97] Kreyszig 1988, p. 667
[98] Fourier 1822
[99] Gasquet & Witomski 1999, p. 67
[100] Ifeachor & Jervis 2002, pp. 3–4, 11
[101] Wallace Feb 1992
[102] Ifeachor & Jervis 2002, p. 132
[103] Gasquet & Witomski 1999, §10.2
[104] Ifeachor & Jervis 2002, pp. 307–310
[105] Gasquet & Witomski 1999, §10.3
[106] Schönhage & Strassen 1971.
[107] That is to say (BSE-3 2001), the plane passing through the point of contact P such that the distance from a point P_1 on the surface to the plane is infinitesimally small compared to the distance from P_1 to P in the limit as P_1 approaches P along the surface.
[108] Spivak 1999, ch. 3
[109] Jost 2005. See also Lorentzian manifold.
[110] Misner, Thorne & Wheeler 1973, ch. 1.8.7, p. 222 and ch. 2.13.5, p. 325
[111] Jost 2005, ch. 3.1
[112] Varadarajan 1974, ch. 4.3, Theorem 4.3.27
[113] Spivak 1999, ch. 3
[114] That is, there is a homeomorphism from $\pi^{-1}(U)$ to $V \times U$ which restricts to linear isomorphisms between fibers.
[115] Kreyszig 1991, §34, p. 108.
[116] A line bundle, such as the tangent bundle of S^1 is trivial if and only if there is a section that vanishes nowhere, see Husemoller 1994, Corollary 8.3. The sections of the tangent bundle are just vector fields.
[117] Eisenberg & Guy 1979
[118] Atiyah 1989.
[119] Artin 1991, ch. 12.
[120] Meyer 2000, Example 5.13.5, p. 436.
[121] Meyer 2000, Exercise 5.13.15–17, p. 442.
[122] Coxeter 1987.
[123] http://www.ams.org/mathscinet-getitem?mr=763890
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Groupoids and Double Groupoids

Groupoid

In mathematics, especially in category theory and homotopy theory, a **groupoid** (less often **Brandt groupoid** or **virtual group**) generalises the notion of group and of category in several equivalent ways. A groupoid can be seen as a:

- *Group* with a partial function replacing the binary operation;
- *Category* in which every morphism is an isomorphism. A category of this sort can be viewed as augmented with a unary operation, called *inverse* by analogy with group theory.

Special cases include:

- *Setoids*, that is: sets which come with an equivalence relation;
- *G-sets*, sets equipped with an action of a group *G*.

Groupoids are often used to reason about geometrical objects such as manifolds. Heinrich Brandt introduced groupoids implicitly via Brandt semigroups in 1926.[1]

Definitions

Algebraic

A groupoid is a set *G* with a unary operation $^{-1}: G \to G,$ and a partial function $*: G \times G \to G.$ * is not a binary operation because it is not necessarily defined for all possible pairs of *G*-elements. The precise conditions under which * is defined are not articulated here and vary by situation.
$*$ and $^{-1}$ have the following axiomatic properties. Let *a*, *b*, and *c* be elements of *G*. Then:

- *Associativity*: If $a * b$ and $b * c$ are defined, then $(a * b) * c$ and $a * (b * c)$ are defined and equal. Conversely, if either of these last two expressions is defined, then so is the other (and again they are equal).
- *Inverse*: $a^{-1} * a$ and $a * a^{-1}$ are always defined.
- *Identity*: If $a * b$ is defined, then $a * b * b^{-1} = a$, and $a^{-1} * a * b = b$. (The previous two axioms already show that these expressions are defined and unambiguous.)

In short:

- $(a * b) * c = a * (b * c)$;
- $(a * b) * b^{-1} = a$;
- $a^{-1} * (a * b) = b$.

From these axioms, two easy and convenient theorems follow:

- $(a^{-1})^{-1} = a$;
- If $a * b$ is defined, then $(a * b)^{-1} = b^{-1} * a^{-1}$.

Category theoretic

A groupoid is a small category in which every morphism is an isomorphism, and hence invertible. More precisely, a groupoid G is:

- A set G_0 of *objects*;
- For each pair of objects x and y in G_0, there exists a (possibly empty) set $G(x,y)$ of *morphisms* (or *arrows*) from x to y. We write $f : x \to y$ to indicate that f is an element of $G(x,y)$.

The objects and morphisms have the properties:

- For every object x, there exists the element id_x of $G(x,x)$;
- For each triple of objects x, y, and z, there exists the function $\mathrm{comp}_{x,y,z} : G(x,y) \times G(y,z) \to G(x,z)$. We write gf for $\mathrm{comp}_{x,y,z}(f,g)$, where $f \in G(x,y)$, and $g \in G(y,z)$;
- There exists the function $\mathrm{inv}_{x,y} : G(x,y) \to G(y,x)$.

Moreover, if $f : x \to y$, $g : y \to z$, and $h : z \to w$, then:

- $f\mathrm{id}_x = f$ and $\mathrm{id}_y f = f$;
- $(hg)f = h(gf)$;
- $f\mathrm{inv}(f) = \mathrm{id}_y$ and $\mathrm{inv}(f)f = \mathrm{id}_x$.

If f is an element of $G(x,y)$ then x is called the **source** of f, written $s(f)$, and y the **target** of f (written $t(f)$).

Comparing the definitions

The algebraic and category-theoretic definitions are equivalent, as follows. Given a groupoid in the category-theoretic sense, let G be the disjoint union of all of the sets $G(x,y)$ (i.e. the sets of morphisms from x to y). Then comp and inv become partially defined operations on G, and inv will in fact be defined everywhere; so we define * to be comp and -1 to be inv. Thus we have a groupoid in the algebraic sense. Explicit reference to G_0 (and hence to id) can be dropped.

Conversely, given a groupoid G in the algebraic sense, with typical element f, let G_0 be the set of all elements of the form $f{*}f^{1}$. In other words, the objects are identified with the identity morphisms, so that id_x is just x. Let $G(x,y)$ be the set of all elements f such that $y\,f\,x$ is defined. Then $^{-1}$ and * break up into several functions on the various $G(x,y)$, which may be called inv and comp, respectively.

Sets in the definitions above may be replaced with classes, as is generally the case in category theory.

Vertex groups

Given a groupoid G, the **vertex groups** or **isotropy group** in G are the subsets of the form $G(x,x)$, where x is any object of G. It follows easily from the axioms above that these are indeed groups, as every pair of elements is composable and inverses are in the same vertex group.

Groupoid Category

The category whose objects are groupoids and whose morphisms are groupoid homomorphisms is called the **groupoid category**, or the **category of groupoids**.

Examples

Linear algebra

Given a field K, the corresponding **general linear groupoid** $GL_*(K)$ consists of all invertible matrices whose entries range over K. Matrix multiplication interprets composition. If $G = GL_*(K)$, then the set of natural numbers is a proper subset of G_0, since for each natural number n, there is a corresponding identity matrix of dimension n. $G(m,n)$ is empty unless $m=n$, in which case it is the set of all nxn invertible matrices.

Topology

Given a topological space X, let G_0 be the set X. The morphisms from the point p to the point q are equivalence classes of continuous paths from p to q, with two paths being equivalent if they are homotopic. Two such morphisms are composed by first following the first path, then the second; the homotopy equivalence guarantees that this composition is associative. This groupoid is called the fundamental groupoid of X, denoted $\pi_1(X)$. The usual fundamental group $\pi_1(X,x)$ is then the vertex group for the point x.

An important extension of this idea is to consider the fundamental groupoid $\pi_1(X,A)$ where A is a set of "base points" and a subset of X. Here, one considers only paths whose endpoints belong to A. $\pi_1(X,A)$ is a sub-groupoid of $\pi_1(X)$. The set A may be chosen according to the geometry of the situation at hand.

Equivalence relation

If X is a set with an equivalence relation denoted by infix $\sim$, then a groupoid "representing" this equivalence relation can be formed as follows:

- The objects of the groupoid are the elements of X;
- For any two elements x and y in X, there is a single morphism from x to y if and only if $x \sim y$.

Group action

If the group G acts on the set X, then we can form the **action groupoid** representing this group action as follows:

- The objects are the elements of X;
- For any two elements x and y in X, there is a morphism from x to y corresponding to every element g of G such that $gx = y$;
- Composition of morphisms interprets the binary operation of G.

More explicitly, the *action groupoid* is the set $G \times X$ with source and target maps $s(g,x) = x$ and $t(g,x) = gx$. It is often denoted $G \ltimes X$ (or $X \rtimes G$). Multiplication (or composition) in the groupoid is then $(h,y)(g,x) = (hg,x)$ which is defined provided $y=gx$.

For x in X, the vertex group consists of those (g,x) with $gx = x$, which is just the isotropy subgroup at x for the given action (which is why vertex groups are also called isotropy groups).

Another way to describe G-sets is the functor category $[\mathrm{Gr}, \mathrm{Set}]$, where Gr is the groupoid (category) with one element and isomorphic to the group G. Indeed, every functor F of this category defines a set $X=F\ (\mathrm{Gr})$ and for every g in G (i.e. for every morphism in Gr) induces a bijection $F_g : X \to X$. The categorical structure of the functor F assures us that F defines a G-action on the set X. The (unique) representable functor $F : \mathrm{Gr} \to \mathrm{Set}$ is the Cayley representation of G. In fact, this functor is isomorphic to $\mathrm{Hom}(\mathrm{Gr}, -)$ and so sends $\mathrm{ob}(\mathrm{Gr})$ to the set $\mathrm{Hom}(\mathrm{Gr}, \mathrm{Gr})$ which is by definition the "set" G and the morphism g of Gr (i.e. the element g of G) to the permutation F_g of the set G. We deduce from the Yoneda embedding that the group G is isomorphic to the group $\{F_g \mid g \in G\}$, a subgroup of the group of permutations of G.

Fifteen puzzle

The symmetries of the Fifteen puzzle form a groupoid (not a group, as not all moves can be composed). This groupoid acts on configurations.

Relation to groups

Group-like structures				
	Totality	**Associativity**	**Identity**	**Inverses**
Group	Yes	Yes	Yes	Yes
Monoid	Yes	Yes	Yes	No
Semigroup	Yes	Yes	No	No
Loop	Yes	No	Yes	Yes
Quasigroup	Yes	No	No	Yes
Magma	Yes	No	No	No
Groupoid	No	Yes	Yes	Yes
Category	No	Yes	Yes	No

If a groupoid has only one object, then the set of its morphisms forms a group. Using the algebraic definition, such a groupoid is literally just a group. Many concepts of group theory generalize to groupoids, with the notion of functor replacing that of group homomorphism.

If x is an object of the groupoid G, then the set of all morphisms from x to x forms a group $G(x)$. If there is a morphism f from x to y, then the groups $G(x)$ and $G(y)$ are isomorphic, with an isomorphism given by the mapping $g \to fgf^{-1}$.

Every connected groupoid (that is, one in which any two objects are connected by at least one morphism) is isomorphic to a groupoid of the following form. Pick a group G and a set (or class) X. Let the objects of the groupoid be the elements of X. For elements x and y of X, let the set of morphisms from x to y be G. Composition of morphisms is the group operation of G. If the groupoid is not connected, then it is isomorphic to a disjoint union of groupoids of the above type (possibly with different groups G for each connected component). Thus any groupoid may be given (up to isomorphism) by a set of ordered pairs (X,G).

Note that the isomorphism described above is not unique, and there is no natural choice. Choosing such an isomorphism for a connected groupoid essentially amounts to picking one object x_0, a group isomorphism h from $G(x_0)$ to G, and for each x other than x_0, a morphism in G from x_0 to x.

In category-theoretic terms, each connected component of a groupoid is equivalent (but not isomorphic) to a groupoid with a single object, that is, a single group. Thus any groupoid is equivalent to a multiset of unrelated groups. In other words, for equivalence instead of isomorphism, one need not specify the sets X, only the groups G.

Consider the examples in the previous section. The general linear groupoid is both equivalent and isomorphic to the disjoint union of the various general linear groups $GL_n(F)$. On the other hand:

- The fundamental groupoid of X is equivalent to the collection of the fundamental groups of each path-connected component of X, but an isomorphism requires specifying the set of points in each component;
- The set X with the equivalence relation $\sim$ is equivalent (as a groupoid) to one copy of the trivial group for each equivalence class, but an isomorphism requires specifying what each equivalence class is:
- The set X equipped with an action of the group G is equivalent (as a groupoid) to one copy of G for each orbit of the action, but an isomorphism requires specifying what set each orbit is.

The collapse of a groupoid into a mere collection of groups loses some information, even from a category-theoretic point of view, because it is not natural. Thus when groupoids arise in terms of other structures, as in the above examples, it can be helpful to maintain the full groupoid. Otherwise, one must choose a way to view each $G(x)$ in terms of a single group, and this choice can be arbitrary. In our example from topology, you would have to make a coherent choice of paths (or equivalence classes of paths) from each point p to each point q in the same path-connected component.

As a more illuminating example, the classification of groupoids with one endomorphism does not reduce to purely group theoretic considerations. This is analogous to the fact that the classification of vector spaces with one endomorphism is nontrivial.

Morphisms of groupoids come in more kinds than those of groups: we have, for example, fibrations, covering morphisms, universal morphisms, and quotient morphisms. Thus a subgroup H of a group G yields an action of G on the set of cosets of H in G and hence a covering morphism p from, say, K to G, where K is a groupoid with vertex groups isomorphic to H. In this way, presentations of the group G can be "lifted" to presentations of the groupoid K, and this is a useful way of obtaining information about presentations of the subgroup H. For further information, see the books by Higgins and by Brown in the References.

Another useful fact is that the category of groupoids, unlike that of groups, is cartesian closed.

Lie groupoids and Lie algebroids

When studying geometrical objects, the arising groupoids often carry some differentiable structure, turning them into Lie groupoids. These can be studied in terms of Lie algebroids, in analogy to the relation between Lie groups and Lie algebras.

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2-groupoid

In mathematics, a **2-group**, or **2-dimensional higher group**, is a certain combination of group and groupoid. The 2-groups are part of a larger hierarchy of *n*-groups. In some of the literature, 2-groups are also called **gr-categories** or **groupal groupoids**.

Definition

A 2-group is a monoidal category G in which every morphism is invertible and every object has a weak inverse. (Here, a *weak inverse* of an object x is an object y such that xy and yx are both isomorphic to the unit object.)

Strict 2-groups

Much of the literature focuses on *strict 2-groups*. A strict 2-group is a *strict* monoidal category in which every morphism is invertible and every object has a strict inverse (so that xy and yx are actually equal to the unit object).

A strict 2-group is a group object in a category of categories; as such, they are also called *groupal categories*. Conversely, a strict 2-group is a category object in the category of groups; as such, they are also called *categorical groups*. They can also be identified with crossed modules, and are most often studied in that form. Thus, 2-groups in general can be seen as a weakening of crossed modules.

Every 2-group is equivalent to a strict 2-group, although this can't be done coherently: it doesn't extend to 2-group homomorphisms.

Properties

Weak inverses can always be assigned coherently: one can define a functor on any 2-group G that assigns a weak inverse to each object and makes that object an adjoint equivalance in the monoidal category G.

Given a bicategory B and an object x of B, there is an *automorphism 2-group* of x in B, written $\mathrm{Aut}_B(x)$. The objects are the automorphisms of x, with multiplication given by composition, and the morphisms are the invertible 2-morphisms between these. If B is a 2-groupoid (so all objects and morphisms are weakly invertible) and x is its only object, then $\mathrm{Aut}_B(x)$ is the only data left in B. Thus, 2-groups may be identified with one-object 2-groupoids, much as groups may be idenitified with one-object groupoids and monoidal categories may be identified with one-object bicategories.

If G is a strict 2-group, then the objects of G form a group, called the *underlying group* of G and written G_0. This will not work for arbitrary 2-groups; however, if one identifies isomorphic objects, then the equivalence classes form a group, called the *fundamental group* of G and written $\pi_1(G)$. (Note that even for a strict 2-group, the fundamental group will only be a quotient group of the underlying group.)

As a monoidal category, any 2-group G has a unit object I_G. The automorphism group of I_G is an abelian group by the Eckmann–Hilton argument, written $\mathrm{Aut}(I_G)$ or $\pi_2(G)$.

The fundamental group of G acts on either side of $\pi_2(G)$, and the associator of G (as a monoidal category) defines an element of the cohomology group $H^3(\pi_1(G),\pi_2(G))$. In fact, 2-groups are classified in this way: given a group π_1, an abelian group π_2, a group action of π_1 on π_2, and an element of $H^3(\pi_1,\pi_2)$, there is a unique (up to equivalence) 2-group G with $\pi_1(G)$ isomorphic to π_1, $\pi_2(G)$ isomorphic to π_2, and the other data corresponding.

The fundamental 2-group

Given a topological space X and a point x in that space, there is a *fundamental 2-group* of X at x, written $\Pi_2(X,x)$. As a monoidal category, the objects are loops at x, with multiplication given by concatenation, and the morphisms are basepoint-preserving homotopies between loops, with these morphisms identified if they are themselves homotopic.

Conversely, given any 2-group G, one can find a unique (up to weak homotopy equivalence) pointed connected space whose fundamental 2-group is G and whose homotopy groups π_n are trivial for $n > 2$. In this way, 2-groups classify pointed connected weak homotopy 2-types. This is a generalisation of the construction of Eilenberg–Mac Lane spaces.

If X is a topological space with basepoint x, then the fundamental group of X at x is the same as the fundamental group of the fundamental 2-group of X at x; that is,

$$\pi_1(X,x) = \pi_1(\Pi_2(X,x)).$$

This fact is the origin of the term "fundamental" in both of its 2-group instances.

Similarly,

$$\pi_2(X,x) = \pi_2(\Pi_2(X,x)).$$

Thus, both the first and second homotopy groups of a space are contained within its fundamental 2-group. As this 2-group also defines an action of $\pi_1(X,x)$ on $\pi_2(X,x)$ and an element of the cohomology group $H^3(\pi_1(X,x),\pi_2(X,x))$, this is precisely the data needed to form the Postnikov tower of X if X is a pointed connected homotopy 2-type.

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External links

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Double groupoid

In mathematics, especially in higher dimensional algebra and homotopy theory, a **double groupoid** generalises the notion of groupoid and of category to a higher dimension.

Definition

A **double groupoid D** is a higher dimensional groupoid involving a relationship for both `horizontal' and `vertical' groupoid structures[1] . (A double groupoid can also be considered as a generalization of certain higher dimensional groups[2] .) The geometry of squares and their compositions leads to a common representation of a *double groupoid* in the following diagram:

$$\mathsf{D} = \begin{tikzcd} S \arrow[r, shift left=3, "s^1"] \arrow[r, shift right=3, "t^1"'] \arrow[d, shift right=3, "s_2"'] \arrow[d, shift left=3, "t_2"] & H \arrow[l] \arrow[d, shift right=3, "s"'] \arrow[d, shift left=3, "t"] \\ V \arrow[u] \arrow[r, shift left=3, "s"] \arrow[r, shift right=3, "t"'] & M \arrow[l] \arrow[u] \end{tikzcd}$$

where **M** is a set of `points', **H** and **V** are, respectively, `horizontal' and `vertical' groupoids, and **S** is a set of `squares' with two compositions. The composition laws for a double groupoid **D** make it also describable as a groupoid internal to the category of groupoids.

Given two groupoids **H**, **V** over a set **M**, there is a double groupoid $\Box(H,V)$ with **H,V** as horizontal and vertical edge groupoids, and squares given by quadruples

$$\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix}$$

for which we assume always that h, h' are in **H**, v, v' are in **V**, and that the initial and final points of these edges match in **M** as suggested by the notation, that is for example sh = sv, th = sv',..., etc. The compositions are to be inherited from those of **H,V**, that is:

$$\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \circ_1 \begin{pmatrix} & h' & \\ w & & w' \\ & h'' & \end{pmatrix} = \begin{pmatrix} & h & \\ vw & & v'w' \\ & h'' & \end{pmatrix}, \begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \circ_2 \begin{pmatrix} & k & \\ v' & & v'' \\ & k' & \end{pmatrix} = \begin{pmatrix} & hk & \\ v & & v'' \\ & h'k' & \end{pmatrix}$$

This construction is the right adjoint to the forgetful functor which takes the double groupoid as above, to the pair of groupoids **H,V** over **M**.

Other related constructions are that of a double groupoid with connection [3] and homotopy double groupoids [4] .

Convolution algebra

A *convolution C*-algebra* of a double groupoid can also be constructed by employing the square diagram **D** of a double groupoid [5] .

Double Groupoid Category

The category whose objects are double groupoids and whose morphisms are double groupoid homomorphisms that are double groupoid diagram (**D**) functors is called the **double groupoid category**, or the **category of double groupoids**.

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Algebraic Topology

Crossed module

In mathematics, and especially in homotopy theory, a **crossed module** consists of groups *G* and *H*, where *G* acts on *H* (which we will write on the left), and a homomorphism of groups

$$d\colon H \longrightarrow G,$$

that is equivariant with respect to the conjugation action of *G* on itself:

$$d(gh) = gd(h)g^{-1}$$

and also satisfies the so-called Peiffer identity:

$$d(h_1)h_2 = h_1h_2h_1^{-1}$$

Examples

Let *N* be a normal subgroup of a group *G*. Then, the inclusion

$$d\colon N \longrightarrow G$$

is a crossed module with the conjugation action of *G* on *N*.

For any group *G*, modules over the group ring are crossed *G*-modules with $d = 0$.

For any group *H*, the homomorphism from *H* to Aut(*H*) sending any element of *H* to the corresponding inner automorphism is a crossed module. Thus we have a kind of `automorphism structure' of a group, rather than just a group of automorphisms.

Given any central extension of groups

$$1 \to A \to H \to G \to 1$$

the onto homomorphism

$$d\colon H \to G$$

together with the action of *G* on *H* defines a crossed module. Thus, central extensions can be seen as special crossed modules. Conversely, a crossed module with surjective boundary defines a central extension.

If (*X*,*A*,*x*) is a pointed pair of topological spaces, then the homotopy boundary

$$d\colon \pi_2(X, A, x) \to \pi_1(A, x)$$

from the second relative homotopy group to the fundamental group, may be given the structure of crossed module. It is a remarkable fact that this functor

$$\Pi\colon \text{(pairs of pointed spaces)} \to \text{(crossed modules)}$$

satisfies a form of the van Kampen theorem, in that it preserves certain colimits. See the article on crossed objects in algebraic topology below. The proof involves the concept of homotopy double groupoid of a pointed pair of spaces.

The result on the crossed module of a pair can also be phrased as: if

$$F \to E \to B$$

is a pointed fibration of spaces, then the induced map of fundamental groups

$$d\colon \pi_1(F) \to \pi_1(E)$$

may be given the structure of crossed module. This example is useful in algebraic K-theory. There are higher dimensional versions of this fact using *n*-cubes of spaces.

These examples suggest that crossed modules may be thought of as "2-dimensional groups". In fact, this idea can be made precise using category theory. It can be shown that a crossed module is essentially the same as a categorical group or 2-group: that is, a group object in the category of categories, or equivalently a category object in the category of groups. While this may sound intimidating, it simply means that the concept of crossed module is one version of the result of blending the concepts of "group" and "category". This equivalence is important in understanding and using even higher dimensional versions of groups.

Classifying space

Any crossed module

$$M = (d \colon H \longrightarrow G)$$

has a *classifying space BM* with the property that its homotopy groups are Coker d , in dimension 1, Ker d in dimension 2, and 0 above 2. It is possible to describe conveniently the homotopy classes of maps from a CW-complex to *BM*. This allows one to prove that (pointed, weak) homotopy 2-types are completely described by crossed modules.

External links

- J. Baez and A. Lauda, Higher-dimensional algebra V: 2-groups [1]
- R. Brown, Groupoids and crossed objects in algebraic topology [1]
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Fundamental groupoid

In mathematics, more specifically algebraic topology, the **fundamental group** or **Poincaré group** (named after Henri Poincaré) is a group associated to any given pointed topological space that provides a way of determining when two paths, starting and ending at a fixed base point, can be continuously deformed into each other. Intuitively, it records information about the basic shape, or *holes*, of the topological space. The fundamental group is the first and simplest of the homotopy groups.

Fundamental groups can be studied using the theory of covering spaces, since a fundamental group coincides with the group of deck transformations of the associated universal covering space. Its abelianisation can be identified with the first homology group of the space. When the topological space is homeomorphic to a simplicial complex, its fundamental group can be described explicitly in terms of generators and relations.

Historically, the concept of fundamental group first emerged in the theory of Riemann surfaces, in the work of Bernhard Riemann, Henri Poincaré and Felix Klein, where it describes the monodromy properties of complex functions, as well as providing a complete topological classification of closed surfaces.

Definition

Let X be an arcwise-connected topological space.

Define a **path** as a continuous function f: $[0,1] \to X$.

Define a **loop** as a closed path, i.e. a loop is a path f with $f(0) = f(1)$.

Define an equivalence relation between two loops f and g by calling them equivalent iff there is a continuous function F: $[0,1]\times[0,1] \to X$, s.t. $F(0, \mathrm{p}) = f(p)$ and $F(1, p) = g(p)$ for all p in $[0, 1]$, and $\mathrm{F(t,0) = f(t) = g(t)}$.

Define the **inverse** of a loop f as a new loop $-f$, s.t. $(-f)(x) = f(1 - x)$.

For two paths f and g s.t. $f(1) = g(0)$, define **addition** as a new path $f + g$, s.t.

$$(f+g)(x) = \begin{cases} f(2x) \text{ if } 0 \leq x \leq 1/2 \\ g(2(x-1/2)) \text{ if } 1/2 < x \leq 1 \end{cases}$$

To define **addition of two loops** f and g, choose a path h from $f(0)$ to $g(0)$. Then define $f * g = f + h + g + (-h)$.

In other words, to add loops f and g, create a new loop by tracing f first, then tracing a path h to the starting point of g, then g, then the same path h back to the starting point of f.

Then the set of all such equivalence classes (i.e. loops in X modulo the equivalence defined above) forms a group with respect to loop additions. This group is called the **fundamental group** of the space X.

Intuition

Start with a space (e.g. a surface), and some point in it, and all the loops both starting and ending at this point — paths that start at this point, wander around and eventually return to the starting point. Two loops can be combined together in an obvious way: travel along the first loop, then along the second. Two loops are considered equivalent if one can be deformed into the other without breaking. The set of all such loops with this method of combining and this equivalence between them is the fundamental group.

For the precise definition, let X be a topological space, and let x_0 be a point of X. We are interested in the set of continuous functions $f : [0,1] \to X$ with the property that $f(0) = x_0 = f(1)$. These functions are called **loops** with **base point** x_0. Any two such loops, say f and g, are considered equivalent if there is a continuous function $h : [0,1] \times [0,1] \to X$ with the property that, for all $0 \leq t \leq 1$, $h(t, 0) = f(t)$, $h(t, 1) = g(t)$ and $h(0, t) = x_0 = h(1, t)$. Such an h is called a **homotopy** from f to g, and the corresponding equivalence classes are called **homotopy classes**.

The product $f * g$ of two loops f and g is defined by setting $(f * g)(t) := f(2t)$ if $0 \le t \le 1/2$ and $(f * g)(t) := g(2t - 1)$ if $1/2 \le t \le 1$. Thus the loop $f * g$ first follows the loop f with "twice the speed" and then follows g with twice the speed. The product of two homotopy classes of loops $[f]$ and $[g]$ is then defined as $[f * g]$, and it can be shown that this product does not depend on the choice of representatives.

With the above product, the set of all homotopy classes of loops with base point x_0 forms the **fundamental group** of X at the point x_0 and is denoted

$$\pi_1(X, x_0),$$

or simply $\pi(X, x_0)$. The identity element is the constant map at the basepoint, and the inverse of a loop f is the loop g defined by $g(t) = f(1 - t)$. That is, g follows f backwards.

Although the fundamental group in general depends on the choice of base point, it turns out that, up to isomorphism, this choice makes no difference so long as the space X is path-connected. For path-connected spaces, therefore, we can write $\pi_1(X)$ instead of $\pi_1(X, x_0)$ without ambiguity whenever we care about the isomorphism class only.

Examples

Trivial fundamental group. In Euclidean space $\mathbf{R}^n$, or any convex subset of $\mathbf{R}^n$, there is only one homotopy class of loops, and the fundamental group is therefore the trivial group with one element. A path-connected space with a trivial fundamental group is said to be simply connected.

Infinite cyclic fundamental group. The circle. Each homotopy class consists of all loops which wind around the circle a given number of times (which can be positive or negative, depending on the direction of winding). The product of a loop which winds around m times and another that winds around n times is a loop which winds around $m + n$ times. So the fundamental group of the circle is isomorphic to $(\mathbb{Z}, +)$, the additive group of integers. This fact can be used to give proofs of the Brouwer fixed point theorem and the Borsuk–Ulam theorem in dimension 2. Since the fundamental group is a homotopy invariant, the theory of the winding number for the complex plane minus one point is the same as for the circle.

Free groups of higher rank: Graphs. Unlike the homology groups and higher homotopy groups associated to a topological space, the fundamental group need not be abelian. For example, the fundamental group of the figure eight is the free group on two letters. More generally, the fundamental group of any graph G is a free group. Here the rank of the free group is equal to $1 - \chi(G)$: one minus the Euler characteristic of G, when G is connected.

Knot theory. A somewhat more sophisticated example of a space with a non-abelian fundamental group is the complement of a trefoil knot in $\mathbf{R}^3$.

Functoriality

If $f : X \to Y$ is a continuous map, $x_0 \in X$ and $y_0 \in Y$ with $f(x_0) = y_0$, then every loop in X with base point x_0 can be composed with f to yield a loop in Y with base point y_0. This operation is compatible with the homotopy equivalence relation and with composition of loops. The resulting group homomorphism, called the induced homomorphism, is written as $\pi(f)$ or, more commonly,

$$f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

We thus obtain a functor from the category of topological spaces with base point to the category of groups.

It turns out that this functor cannot distinguish maps which are homotopic relative to the base point: if f and $g : X \to Y$ are continuous maps with $f(x_0) = g(x_0) = y_0$, and f and g are homotopic relative to $\{x_0\}$, then $f_* = g_*$. As a consequence, two homotopy equivalent path-connected spaces have isomorphic fundamental groups:

$$X \simeq Y \Rightarrow \pi_1(X, x_0) \cong \pi_1(Y, y_0).$$

The fundamental group functor takes products to products and coproducts to coproducts. That is, if X and Y are path connected, then

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

and

$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y).$$

(In the latter formula, $\vee$ denotes the wedge sum of topological spaces, and * the free product of groups.) Both formulas generalize to arbitrary products. Furthermore the latter formula is a special case of the Seifert–van Kampen theorem which states that the fundamental group functor takes pushouts along inclusions to pushouts.

Fibrations

A generalization of a product of spaces is given by a fibration,

$$F \to E \to B.$$

Here the total space *E* is a sort of "twisted product" of the base space *B* and the fiber *F*. In general the fundamental groups of *B*, *E* and *F* are terms in a long exact sequence involving higher homotopy groups. When all the spaces are connected, this has the following consequences for the fundamental groups:

- $\pi_1(B)$ and $\pi_1(E)$ are isomorphic if *F* is simply connected
- $\pi_{n+1}(B)$ and $\pi_n(F)$ are isomorphic if *E* is contractible

The latter is often applied to the situation *E* = path space of *B*, *F* = loop space of *B* or *B* = classifying space *BG* of a topological group *G*, *B* = universal *G*-bundle *EG*.

Relationship to first homology group

The fundamental groups of a topological space *X* are related to its first singular homology group, because a loop is also a singular 1-cycle. Mapping the homotopy class of each loop at a base point x_0 to the homology class of the loop gives a homomorphism from the fundamental group $\pi_1(X, x_0)$ to the homology group $H_1(X)$. If *X* is path-connected, then this homomorphism is surjective and its kernel is the commutator subgroup of $\pi_1(X, x_0)$, and $H_1(X)$ is therefore isomorphic to the abelianization of $\pi_1(X, x_0)$. This is a special case of the Hurewicz theorem of algebraic topology.

Universal covering space

If *X* is a topological space that is path connected, locally path connected and locally simply connected, then it has a simply connected universal covering space on which the fundamental group $\pi(X,x_0)$ acts freely by deck transformations with quotient space *X*. This space can be constructed analogously to the fundamental group by taking pairs (x, γ), where *x* is a point in *X* and γ is a homotopy class of paths from x_0 to *x* and the action of $\pi(X, x_0)$ is by concatenation of paths. It is uniquely determined as a covering space.

Examples

Let *G* be a connected, simply connected compact Lie group, for example the special unitary group SU_n, and let Γ be a finite subgroup of *G*. Then the homogeneous space $X = G/\Gamma$ has fundamental group Γ, which acts by right multiplication on the universal covering space *G*. Among the many variants of this construction, one of the most important is given by locally symmetric spaces $X = \Gamma\backslash G/K$, where

- *G* is a non-compact simply connected, connected Lie group (often semisimple),
- *K* is a maximal compact subgroup of *G*
- Γ is a discrete countable torsion-free subgroup of *G*.

In this case the fundamental group is Γ and the universal covering space *G*/*K* is actually contractible (by the Cartan decomposition for Lie groups).

As an example take $G = SL_2(\mathbf{R})$, $K = SO_2$ and Γ any torsion-free congruence subgroup of the modular group $SL_2(\mathbf{Z})$.

An even simpler example is given by $G = \mathbf{R}$ (so that K is trivial) and $\Gamma = \mathbf{Z}$: in this case $X=\mathbf{R}/\mathbf{Z} = S^1$.

From the explicit realization, it also follows that the universal covering space of a path connected topological group H is again a path connected topological group G. Moreover the covering map is a continuous open homomorphism of G onto H with kernel Γ, a closed discrete normal subgroup of G:

$$1 \to \Gamma \to G \to H \to 1.$$

Since G is a connected group with a continuous action by conjugation on a discrete group Γ, it must act trivially, so that Γ has to be a subgroup of the center of G. In particular $\pi_1(H) = \Gamma$ is an Abelian group; this can also easily be seen directly without using covering spaces. The group G is called the *universal covering group* of H.

As the universal covering group suggests, there is an analogy between the fundamental group of a topological group and the center of a group; this is elaborated at Lattice of covering groups.

Edge-path group of a simplicial complex

If X is a connected simplicial complex, an *edge-path* in X is defined to be a chain of vertices connected by edges in X. Two edge-paths are said to be *edge-equivalent* if one can be obtained from the other by successively switching between an edge and the two opposite edges of a triangle in X. If v is a fixed vertex in X, an *edge-loop* at v is an edge-path starting and ending at v. The **edge-path group** $E(X, v)$ is defined to be the set of edge-equivalence classes of edge-loops at v, with product and inverse defined by concatenation and reversal of edge-loops.

The edge-path group is naturally isomorphic to $\pi_1(|X|, v)$, the fundamental group of the geometric realisation $|X|$ of X. Since it depends only on the 2-skeleton X^2 of X (i.e. the vertices, edges and triangles of X), the groups $\pi_1(|X|,v)$ and $\pi_1(|X^2|, v)$ are isomorphic.

The edge-path group can be described explicitly in terms of generators and relations. If T is a maximal spanning tree in the 1-skeleton of X, then $E(X, v)$ is canonically isomorphic to the group with generators the oriented edges of X not occurring in T and relations the edge-equivalences corresponding to triangles in X containing one or more edge not in T. A similar result holds if T is replaced by any simply connected—in particular contractible—subcomplex of X. This often gives a practical way of computing fundamental groups and can be used to show that every finitely presented group arises as the fundamental group of a finite simplicial complex. It is also one of the classical methods used for topological surfaces, which are classified by their fundamental groups.

The *universal covering space* of a finite connected simplicial complex X can also be described directly as a simplicial complex using edge-paths. Its vertices are pairs (w,γ) where w is a vertex of X and γ is an edge-equivalence class of paths from v to w. The k-simplices containing (w,γ) correspond naturally to the k-simplices containing w. Each new vertex u of the k-simplex gives an edge wu and hence, by concatenation, a new path γ_u from v to u. The points (w,γ) and (u, γ_u) are the vertices of the "transported" simplex in the universal covering space. The edge-path group acts naturally by concatenation, preserving the simplicial structure, and the quotient space is just X.

It is well-known that this method can also be used to compute the fundamental group of an arbitrary topological space. This was doubtless known to Čech and Leray and explicitly appeared as a remark in a paper by Weil (1960); various other authors such as L. Calabi, W-T. Wu and N. Berikashvili have also published proofs. In the simplest case of a compact space X with a finite open covering in which all non-empty finite intersections of open sets in the covering are contractible, the fundamental group can be identified with the edge-path group of the simplicial complex corresponding to the nerve of the covering.

Realizability

- Every group can be realized as the fundamental group of a connected CW-complex of dimension 2 (or higher). As noted above, though, only free groups can occur as fundamental groups of 1-dimensional CW-complexes (that is, graphs).
- Every finitely presented group can be realized as the fundamental group of a compact, connected, smooth manifold of dimension 4 (or higher). But there are severe restrictions on which groups occur as fundamental groups of low-dimensional manifolds. For example, no free abelian group of rank 4 or higher can be realized as the fundamental group of a manifold of dimension 3 or less.

Related concepts

The fundamental group measures the 1-dimensional hole structure of a space. For studying "higher-dimensional holes", the homotopy groups are used. The elements of the n-th homotopy group of X are homotopy classes of (basepoint-preserving) maps from S^n to X.

The set of loops at a particular base point can be studied without regarding homotopic loops as equivalent. This larger object is the loop space.

For topological groups, a different group multiplication may be assigned to the set of loops in the space, with pointwise multiplication rather than concatenation. The resulting group is the loop group.

Fundamental groupoid

Rather than singling out one point and considering the loops based at that point up to homotopy, one can also consider *all* paths in the space up to homotopy (fixing the initial and final point). This yields not a group but a groupoid, the **fundamental groupoid** of the space.

See also

- Homotopy group, generalization of fundamental group

There are also similar notions of fundamental group for algebraic varieties (the étale fundamental group) and for orbifolds (the orbifold fundamental group).

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External links

- Dylan G.L. Allegretti, *Simplicial Sets and van Kampen's Theorem* [3] *(An elementary discussion of the fundamental groupoid of a topological space and the fundamental groupoid of a simplicial set).*
- Animations to introduce to the fundamental group by Nicolas Delanoue [4]

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Seifert–van Kampen theorem

In mathematics, the **Seifert-van Kampen theorem** of algebraic topology, sometimes just called **van Kampen's theorem**, expresses the structure of the fundamental group of a topological space X, in terms of the fundamental groups of two open, path-connected subspaces U and V that cover X. It can therefore be used for computations of the fundamental group of spaces that are constructed out of simpler ones.

The underlying idea is that paths in X can be partitioned into journeys: through the intersection W of U and V, through U but outside V, and through V outside U. In order to move segments of paths around, by homotopy to form loops returning to a base point w in W, we should assume U, V and W are path-connected and that W isn't empty. We also assume that U and V are open subspaces with union X.

Under these conditions, $\pi_1(U,w)$, $\pi_1(V,w)$, and $\pi_1(W,w)$, together with homomorphisms induced by the inclusion maps of W into U and V:

$$I\colon \pi_1(W,w) \to \pi_1(U,w)$$

and

$$J\colon \pi_1(W,w) \to \pi_1(V,w)$$

are sufficient data to determine $\pi_1(X,w)$. The maps I and J extend to an epimorphism:

$$\Phi\colon \pi_1(U,w) * \pi_1(V,w) \to \pi_1(X,w)$$

where $\pi_1(U,w) * \pi_1(V,w)$ is the free product of $\pi_1(U,w)$ and $\pi_1(V,w)$. The kernel of the map Φ are the loops in W that, when viewed in X, are homotopic to the trivial one at w. The group $\pi_1(X,w)$ is therefore isomorphic to $\pi_1(U,w) * \pi_1(V,w)$ modulo such elements, more precisely, to the amalgamated free product $\pi_1(U,w) *_{\pi(W,w)} \pi_1(V,w)$. In particular, when W is simply connected (so that its fundamental group is the trivial group), the theorem says that $\pi_1(X,w)$ is isomorphic to the free product $\pi_1(U,w) * \pi_1(V,w)$.

Equivalent formulations

In the language of combinatorial group theory, $\pi_1(X,w)$ is the free product with amalgamation of $\pi_1(U,w)$ and $\pi_1(V,w)$, with respect to the (not necessarily injective) homomorphisms I and J. Given group presentations:

$$\pi_1(U,w) = \langle u_1, \ldots, u_k | \alpha_1, \ldots, \alpha_l \rangle$$

$$\pi_1(V,w) = \langle v_1, \ldots, v_m | \beta_1, \ldots, \beta_n \rangle \text{ , and}$$

$$\pi_1(W,w) = \langle w_1, \ldots, w_p | \gamma_1, \ldots, \gamma_q \rangle$$

the amalgamation can be presented as

$$\pi_1(X,w) = \langle u_1, \ldots, u_k, v_1, \ldots, v_m | \alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_n, I(w_1)J(w_1)^{-1}, \ldots, I(w_p)J(w_p)^{-1} \rangle \text{ .}$$

In category theory, $\pi_1(X,w)$ is the pushout of the diagram:

$$\pi_1(U,w) \leftarrow \pi_1(W,w) \rightarrow \pi_1(V,w).$$

Van Kampen's theorem for fundamental groups

Van Kampen's theorem for fundamental groups[1] :

Let X be a topological space which is the union of the interiors of two path connected subspaces $X_1,\ X_2$. *Suppose* $X_0 := X_1 \cap X_2$ *is path connected. Let also* $* \in X_0$ *and* $i_k : \pi_1(X_0,*) \to \pi_1(X_k,*)$, $j_k : \pi_1(X_k,*) \to \pi_1(X,*)$ *be induced by the inclusions for k= 1, 2. Then X is path connected and the natural morphism* $\pi_1(X_1,*) *_{\pi_1(X_0,*)} \pi_1(X_2,*) \to \pi_1(X,*)$ *is an isomorphism*, that is, *the fundamental group of X is the free product of the fundamental groups of* X_1 *and* X_2 *with amalgamation* of $\pi_1(X_0,*)$.

Usually the morphisms induced by inclusion in this theorem are not themselves injective, and the more precise version of the statement is in terms of pushouts of groups. The notion of pushout in the category of groupoids allows for a version of the theorem for the non path connected case, using the fundamental groupoid $\pi_1(X,A)$ on a set A of base points,[2] . *This groupoid consists of homotopy classes relative to the end points of paths in X joining points of* $A \cap X$. *In particular, if X is a contractible space, and A consists of two distinct points of X, then* $\pi_1(X,A)$ *is easily seen to be isomorphic to the groupoid often written* $\mathcal{I}$ *with two vertices and exactly one morphism between any two vertices. This groupoid plays a role in the theory of groupoids analogous to that of the group of integers in the theory of groups*[3] .

Theorem: *Let the topological space X be covered by the interiors of two subspaces* X_1, X_2 *and let A be a set which meets each path component of* X_1, X_2 *and* $X_0 := X_1 \cap X_2$. *Then A meets each path component of X and the diagram* ***P*** *of morphisms induced by inclusion*

$$\begin{tikzcd}
\pi_1(X_0,A) \arrow[r, "\pi_1(i_1)"] \arrow[d, "\pi_1(i_2)"'] & \pi_1(X_1,A) \arrow[d, "\pi_1(j_1)"] \\
\pi_1(X_2,A) \arrow[r, "\pi_1(j_2)"'] & \pi_1(X,A)
\end{tikzcd}$$

is a pushout diagram in the category of groupoids.[4]

The interpretation of this theorem as a calculational tool for fundamental groups needs some development of `combinatorial groupoid theory',[5] [6] . This theorem implies the calculation of the fundamental group of the circle as the group of integers, since the group of integers is obtained from the groupoid $\mathcal{I}$ by identifying, in the category of groupoids, its two vertices.

There is a version of the last theorem when X is covered by the union of the interiors of a family $\{U_\lambda : \lambda \in \Lambda\}$ of subsets[7] [8] . The conclusion is that if A meets each path component of all 1,2,3-fold intersections of the sets U_λ , then A meets all path components of X and the diagram $\bigsqcup_{(\lambda,\mu)\in\Lambda^2} \pi_1(U_\lambda \cap U_\mu, A) \rightrightarrows \bigsqcup_{\lambda\in\Lambda} \pi_1(U_\lambda, A) \to \pi_1(X,A)$ of morphisms induced by inclusions is a coequaliser in the category of groupoids.

Examples

One can use Van Kampen's theorem to calculate fundamental groups for topological spaces that can be decomposed into simpler spaces. For example, consider the sphere S^2. Pick open sets $A = S^2 - n$ and $B = S^2 - s$ where n and s denote the north and south poles respectively. Then we have the property that A, B and A B are open path connected sets. Thus we can see that there is a commutative diagram including A B into A and B and then another inclusion from A and B into S^2 and that there is a corresponding diagram of homomorphisms between the

fundamental groups of each subspace. Applying Van Kampen's theorem gives the result $\pi_1(S^2) = \pi_1(A) * \pi_1(B)/ker(\Phi)$. Ho
homeomorphic to $\mathbf{R}^2$ which is simply connected, so both A and B have trivial fundamental groups. It is clear from this that the fundamental group of S^2 is trivial.

A more complicated example is the calculation of the fundamental group of a genus *n* orientable surface *S*, otherwise known as the *genus n surface group*. One can construct *S* using its standard fundamental polygon. For the first open set *A*, pick a disk within the center of the polygon. Pick *B* to be the complement in *S* of the center point of *A*. Then the intersection of *A* and *B* is an annulus, which is known to be homotopy equivalent to (and so has the same fundamental group as) a circle. Then $\pi_1(A \cap B) = \pi_1(S^1)$, which is the integers, and $\pi_1(A) = \pi_1(D^2) = 1$. Thus the inclusion of $\pi_1(A \cap B)$ into $\pi_1(A)$ sends any generator to the trivial element. However, the inclusion of $\pi_1(A \cap B)$ into $\pi_1(B)$ is not trivial. In order to understand this, first one must calculate $\pi_1(B)$. This is easily done as one can deformation retract B (which is *S* with one point deleted) onto the edges labeled by $A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2^{-1}B_2^{-1} \ldots A_nB_nA_n^{-1}B_n^{-1}$. This space is known to be the wedge sum of 2*n* circles (also called a bouquet of circles), which further is known to have fundamental group isomorphic to the free group with 2*n* generators, which in this case can be represented by the edges themselves: $\{A_1, B_1, \ldots, A_n, B_n\}$. We now have enough information to apply Van Kampen's theorem. The generators are the loops $\{A_1, B_1, \ldots, A_n, B_n\}$ (*A* is simply connected, so it contributes no generators) and there is exactly one relation: $A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2^{-1}B_2^{-1} \ldots A_nB_nA_n^{-1}B_n^{-1} = 1$. Using generators and relations, this group is denoted $\langle A_1, B_1, \ldots, A_n, B_n | A_1B_1A_1^{-1}B_1^{-1} \ldots A_nB_nA_n^{-1}B_n^{-1} \rangle$.

Generalizations

This theorem has been extended to the non-connected case by using the fundamental groupoid $_1$(X,A) on a *set A of base points*, which consists of homotopy classes of paths in X joining points of X which lie in A. The connectivity conditions for the theorem then become that A meets each path-component of U,V,W. The pushout is now in the category of groupoids. This extended theorem allows the determination of the fundamental group of the circle, and many other useful cases. For example, if the intersection W has two path components, it is convenient to let A consist of one point in each of these components. A theorem for arbitrary covers, with the restriction that A meets all three fold intersections of the sets of the cover, is given in the paper by Brown and Razak cited below. Applications of the fundamental groupoid on a set of base points to the Jordan curve theorem, Covering space, and orbit space are given in Ronald Brown's book cited below.

In the case of orbit spaces, it is convenient to take A to include all the fixed points of the action. An example here is the conjugation action on the circle.

The version that allows more than two overlapping sets but with A a singleton is also given in Allen Hatcher's book below, theorem 1.20.

In fact, we can extend van Kampen's theorem significantly further by considering the fundamental groupoid $\prod(X)$, a small category whose objects are points of X and whose arrows are paths between points modulo homotopy equivalence. In this case, to determine the fundamental groupoid of a space, we need only know the fundamental groupoids of the open sets covering X as follows: create a new category in which the objects are fundamental groupoids of the open sets, with an arrow between groupoids if the domain space is a subspace of the codomain. Then van Kampen's theorem is the assertion that the fundamental groupoid of X is the colimit of the diagram. For details, see Peter May's book, chapter 2.

References to higher dimensional versions of the theorem which yield some information on homotopy types are given in an article on higher dimensional group theories and groupoids.[9]

See also

- Higher dimensional algebra
- Higher category theory
- Alexander Grothendieck
- Van Kampen
- Ronald Brown (mathematician)

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Lie groupoid

In mathematics, a **Lie groupoid** is a groupoid where the set Ob of objects and the set Mor of morphisms are both manifolds, the source and target operations

$$s, t : Mor \to Ob$$

are submersions, and all the category operations (source and target, composition, and identity-assigning map) are smooth.

A Lie groupoid can thus be thought of as a "many-object generalization" of a Lie group, just as a groupoid is a many-object generalization of a group. Just as every Lie group has a Lie algebra, every Lie groupoid has a Lie algebroid.

Examples

- Any Lie group gives a Lie groupoid with one object, and conversely. So, the theory of Lie groupoids includes the theory of Lie groups.
- Given any manifold M , there is a Lie groupoid called the pair groupoid, with M as the manifold of objects, and precisely one morphism from any object to any other. In this Lie groupoid the manifold of morphisms is thus $M \times M$.
- Given a Lie group G acting on a manifold M , there is a Lie groupoid called the translation groupoid with one morphism for each triple $g \in G, x, y \in M$ with $gx = y$.
- Any foliation gives a Lie groupoid.
- Any principal bundle $P \to M$ with structure group *G* gives a groupoid, namely $P \times P/G$ over *M*, where *G* acts on the pairs componentwise. Composition is defined via compatible representatives as in the pair groupoid.

Morita Morphisms and Smooth Stacks

Beside isomorphism of groupoids there is a more coarse notation of equivalence, the so called Morita equivalence. A quite general example is the Morita-morphism of the **Čech groupoid** which goes as follows. Let *M* be a smooth manifold and $\{U_\alpha\}$ an open cover of *M*. Define $G_0 := \bigsqcup_\alpha U_\alpha$ the disjoint union with the obvious submersion $p : G_0 \to M$. In order to encode the structure of the manifold *M* define the set of morphisms $G_1 := \bigsqcup_{\alpha,\beta} U_{\alpha\beta}$ where $U_{\alpha\beta} = U_\alpha \cap U_\beta \subset M$. The source and target map are defined as the embeddings $s : U_{\alpha\beta} \to U_\alpha$ and $t : U_{\alpha\beta} \to U_\beta$. And multiplication is the obvious one if we read the $U_{\alpha\beta}$ as subsets of *M* (compatible points in $U_{\alpha\beta}$ and $U_{\beta\gamma}$ actually are the same in *M* and also lie in $U_{\alpha\gamma}$).
This Čech groupoid is in fact the pullback groupoid of $M \rightrightarrows M$, i.e. the trivial groupoid over *M*, under *p*. That is what makes it Morita-morphism.

In order to get the notion of an equivalence relation we need to make the construction symmetric and show that it is also transitive. In this sense we say that 2 groupoids $G_1 \rightrightarrows G_0$ and $H_1 \rightrightarrows H_0$ are Morita equivalent iff there exists a third groupoid $K_1 \rightrightarrows K_0$ together with 2 Morita morphisms from *G* to *K* and *H* to *K*. Transitivity is an interesting construction in the category of groupoid principal bundles and left to the reader.
It arises the question of what is preserved under the Morita equivalence. There are 2 obvious things, one the coarse quotient/ orbit space of the groupoid $G_0/G_1 = H_0/H_1$ and secondly the stabilizer groups $G_p \cong H_q$ for corresponding points $p \in G_0$ and $q \in H_0$.
The further question of what is the structure of the coarse quotient space leads to the notion of a smooth stack. We can expect the coarse quotient to be a smooth manifold if for example the stabilizer groups are trivial (as in the example of the Čech groupoid). But if the stabilizer groups change we cannot expect a smooth manifold any longer. The solution is to revert the problem and to define:

A **smooth stack** is a Morita-equivalence class of Lie groupoids. The natural geometric objects living on the stack are the geometric objects on Lie groupoids invariant under Morita-equivalence. As an example consider the Lie groupoid cohomology.

Examples

- The notion of smooth stack is quite general, obviously all smooth manifolds are smooth stacks.
- But also orbifolds are smooth stacks, namely (equivalence classes of) étale groupoids.
- Orbit spaces of foliations are another class of examples

External links

Alan Weinstein, Groupoids: unifying internal and external symmetry, *AMS Notices*, **43** (1996), 744-752. Also available as arXiv:math/9602220 [1]

Kirill Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, Cambridge U. Press, 1987.

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[1] http://arxiv.org/abs/math/9602220

Lie algebroid

In mathematics, Lie algebroids serve the same role in the theory of Lie groupoids that Lie algebras serve in the theory of Lie groups: reducing global problems to infinitesimal ones. Just as a Lie groupoid can be thought of as a "Lie group with many objects", a Lie algebroid is like a "Lie algebra with many objects".

More precisely, a **Lie algebroid** is a triple $(E, [\cdot,\cdot], \rho)$ consisting of a vector bundle E over a manifold M, together with a Lie bracket $[\cdot,\cdot]$ on its module of sections $\Gamma(E)$ and a morphism of vector bundles $\rho : E \rightarrow TM$ called the **anchor**. Here TM is the tangent bundle of M. The anchor and the bracket are to satisfy the Leibniz rule:

$$[X, fY] = \rho(X)f \cdot Y + f[X, Y]$$

where $X, Y \in \Gamma(E), f \in C^\infty(M)$ and $\rho(X)f$ is the derivative of f along the vector field $\rho(X)$. It follows that

$$\rho([X, Y]) = [\rho(X), \rho(Y)]$$

for all $X, Y \in \Gamma(E)$.

Examples

- Every Lie algebra is a Lie algebroid over the one point manifold.
- The tangent bundle TM of a manifold M is a Lie algebroid for the Lie bracket of vector fields and the identity of TM as an anchor.
- Every integrable subbundle of the tangent bundle — that is, one whose sections are closed under the Lie bracket — also defines a Lie algebroid.
- Every bundle of Lie algebras over a smooth manifold defines a Lie algebroid where the Lie bracket is defined pointwise and the anchor map is equal to zero.
- To every Lie groupoid is associated a Lie algebroid, generalizing how a Lie algebra is associated to a Lie group (see also below). For example, the Lie algebroid TM comes from the pair groupoid whose objects are M, with one isomorphism between each pair of objects. Unfortunately, going back from a Lie algebroid to a Lie groupoid is not always possible[1] , but every Lie algebroid gives a stacky Lie groupoid[2] [3] .
- Given the action of a Lie algebra g on a manifold M, the set of g -invariant vector fields on M is a Lie algebroid over the space of orbits of the action.
- **Atiyah algebroid**. Given a vector bundle V over a smooth manifold M consider its derivations, i.e. smooth $\mathbb{R}$ -linear maps $\psi : \Gamma(V) \rightarrow \Gamma(V)$ for which exists a vector field *X* such that they fulfill the Leibniz rule $\psi(fv) = X[f]v + f\psi(v)$ for all smooth functions *f* and all sections *v* into the vector bundle. The association $\psi \rightarrow X$ is clearly linear and thus comes from a map of vector bundles $\rho : A(V) \rightarrow TM$ (if you find the bundle whose sections give the derivations). The Atiyah algebroid is further characterized by fitting into the following short exact sequence: $0 \rightarrow \mathrm{End}_M(V) \rightarrow A(V) \rightarrow TM \rightarrow 0$ To see that the Atiyah algebroid exists for every vector bundle note that it is the Lie algebroid associated to the Lie groupoid coming from the frame bundle of the vector bundle *V*.

Lie algebroid associated to a Lie groupoid

To describe the construction let us fix some notation. *G* is the space of morphisms of the Lie groupoid, *M* the space of objects, $e : M \to G$ the units and $t : G \to M$ the target map.

$T^tG = \bigcup_{p \in M} T(t^{-1}(p)) \subset TG$ the *t*-fiber tangent space. The Lie algebroid is now the vector bundle $A := e^*T^tG$. This inherits a bracket from *G*, because we can identify the *M*-sections into *A* with left-invariant vector fields on *G*. Further these sections act on the smooth functions of *M* by identifying these with left-invariant functions on *G*.

As a more explicit example consider the Lie algebroid associated to the pair groupoid $G := M \times M$. The target map is $t : G \to M : (p, q) \mapsto p$ and the units $e : M \to G : p \mapsto (p, p)$. The *t*-fibers are $p \times M$ and therefore $T^tG = \bigcup_{p \in M} p \times TM \subset TM \times TM$. So the Lie algebroid is the vector bundle $A := e^*T^tG = \bigcup_{p \in M} T_pM = TM$. The extension of sections *X* into *A* to left-invariant vector fields on *G* is simply $\tilde{X}(p, q) = 0 \oplus X(q)$ and the extension of a smooth function *f* from *M* to a left-invariant function on *G* is $\tilde{f}(p, q) = f(q)$. Therefore the bracket on *A* is just the Lie bracket of tangent vector fields and the anchor map is just the identity.

Of course you could do an analog construction with the source map and right-invariant vector fields/ functions. However you get an isomorphic Lie algebroid, with the explicit isomorphism i_*, where $i : G \to G$ is the inverse map.

External links

- Alan Weinstein, Groupoids: unifying internal and external

symmetry, *AMS Notices*, **43** (1996), 744-752. Also available as arXiv:math/9602220 [1]

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Chain complex

In mathematics, **chain complex** and **cochain complex** are constructs originally used in the field of algebraic topology. They are algebraic means of representing the relationships between the cycles and boundaries in various dimensions of some "space". Here the "space" could be a topological space or an algebraic construction such as a simplicial complex. More generally, homological algebra includes the study of chain complexes in the abstract, without any reference to an underlying space. In this case, chain complexes are studied axiomatically as algebraic structures.

Applications of chain complexes usually define and apply their homology groups (cohomology groups for cochain complexes); in more abstract settings various equivalence relations are applied to complexes (for example starting with the *chain homotopy* idea). Chain complexes are easily defined in abelian categories, also.

Formal definition

A **chain complex** $(A_\bullet, d_\bullet)$ is a sequence of abelian groups or modules ... A_2, A_1, A_0, A_{-1}, A_{-2}, ... connected by homomorphisms (called **boundary operators**) $d_n : A_n \to A_{n-1}$, such that the composition of any two consecutive maps is zero: $d_n \circ d_{n+1} = 0$ for all n. They are usually written out as:

$$\cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} A_{n-2} \to \cdots \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \xrightarrow{d_{-1}} A_{-2} \xrightarrow{d_{-2}} \cdots$$

A variant on the concept of chain complex is that of *cochain complex*. A **cochain complex** $(A^\bullet, d^\bullet)$ is a sequence of abelian groups or modules ..., A^{-2}, A^{-1}, A^0, A^1, A^2, ... connected by homomorphisms $d^n : A^n \to A^{n+1}$ such that the composition of any two consecutive maps is zero: $d^{n+1} d^n = 0$ for all n:

$$\cdots \to A^{-2} \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \to \cdots \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \to \cdots.$$

The idea is basically the same. In either case, the index n in A_n resp. A^n is referred to as the **degree** (or **dimension**).

A **bounded chain complex** is one in which almost all the A_i are 0; *i.e.*, a finite complex extended to the left and right by 0's. An example is the complex defining the homology theory of a (finite) simplicial complex. A chain complex is **bounded above** if all degrees above some fixed degree N are 0, and is **bounded below** if all degrees below some fixed degree are 0. Clearly, a complex is bounded above and below iff the complex is bounded.

Fundamental terminology

Leaving out the indices, the basic relation on d can be thought of as

$$dd = 0.$$

The elements of the individual groups of a chain complex are called **chains** (or **cochains** in the case of a cochain complex.) The image of d is the group of **boundaries**, or in a cochain complex, **coboundaries**. The kernel of d (i.e., the subgroup sent to 0 by d) is the group of **cycles**, or in the case of a cochain complex, **cocycles**. From the basic relation, the (co)boundaries lie inside the (co)cycles. This phenomenon is studied in a systematic way using (co)homology groups.

Examples

Singular homology

Suppose we are given a topological space *X*.

Define $C_n(X)$ for natural *n* to be the free abelian group formally generated by singular n-simplices in *X*, and define the boundary map

$$\partial_n : C_n(X) \to C_{n-1}(X) : (\sigma : [v_0, \ldots, v_n] \to X) \mapsto (\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma([v_0, \ldots, \hat{v}_i, \ldots, v_n]),$$

where the hat denotes the omission of a vertex. That is, the boundary of a singular simplex is alternating sum of restrictions to its faces. It can be shown $\partial^2 = 0$, so $(C_\bullet, \partial_\bullet)$ is a chain complex; the *singular homology* $H_\bullet(X)$ is the homology of this complex; that is,

$$H_n(X) = \ker \partial_n / \mathrm{im}\ \partial_{n+1}.$$

de Rham cohomology

The differential *k*-forms on any smooth manifold *M* form an abelian group (in fact an **R**-vector space) called $\Omega^k(M)$ under addition. The exterior derivative d_k maps $\Omega^k(M)$ to $\Omega^{k+1}(M)$, and $d^2 = 0$ follows essentially from symmetry of second derivatives, so the vector spaces of *k*-forms along with the exterior derivative are a cochain complex:

$$\Omega^0(M) \stackrel{d_0}{\to} \Omega^1(M) \to \Omega^2(M) \to \Omega^3(M) \to \cdots.$$

The homology of this complex is the *de Rham cohomology*

$$H^0_{\mathrm{DR}}(M, F) = \ker d_0 = \{\text{locally constant functions on } M \text{ with values in } F\} \cong F^{\#\{\text{connected pieces of } M\}}$$
$$H^k_{\mathrm{DR}}(M) = \ker d_k / \mathrm{im}\, d_{k-1}.$$

Chain maps

A *chain map* *f* between two chain complexes $(A_\bullet, d_{A,\bullet})$ and $(B_\bullet, d_{B,\bullet})$ is a sequence $f_\bullet$ of module homomorphisms $f_n : A_n \to B_n$ for each *n* that intertwines with the differentials on the two chain complexes: $d_{B,n} \circ f_n = f_{n-1} \circ d_{A,n}$. Such a map sends cycles to cycles and boundaries to boundaries, and thus descends to a map on homology: $(f_n)_* : H_\bullet(A_\bullet, d_{A,\bullet}) \to H_\bullet(B_\bullet, d_{B,\bullet})$.

A continuous map of topological spaces induces chain maps in both the singular and de Rham chain complexes described above (and in general for the chain complex defining any homology theory of topological spaces) and thus a continuous map induces a map on homology. Because the map induced on a composition of maps is the composition of the induced maps, these homology theories are functors from the category of topological spaces with continuous maps to the category of abelian groups with group homomorphisms.

It is worth noticing that the concept of chain map reduces to the one of boundary through the construction of the cone of a chain map.

Chain homotopy

Chain homotopies give an important equivalence relation between chain maps. Chain homotopic chain maps induce the same maps on homology groups. A particular case is that homotopic maps between two spaces *X* and *Y* induce the same maps from homology of *X* to homology of *Y*. Chain homotopies have a geometric interpretation; it is described, for example, in the book of Bott and Tu. See Homotopy category of chain complexes for further information.

See also

- Homology
- Differential graded algebra

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Homology

In mathematics (especially algebraic topology and abstract algebra), **homology** (in Greek ὁμός *homos* "identical") is a certain general procedure to associate a sequence of abelian groups or modules with a given mathematical object such as a topological space or a group. See homology theory for more background, or singular homology for a concrete version for topological spaces, or group cohomology for a concrete version for groups.

For a topological space, the homology groups are generally much easier to compute than the homotopy groups, and consequently one usually will have an easier time working with homology to aid in the classification of spaces.

Construction of homology groups

The procedure works as follows. Given an object such as a topological space X, one first defines a *chain complex* $C(X)$ encoding information about X. A chain complex is a sequence of abelian groups or modules $C_0, C_1, C_2, \ldots$ connected by homomorphisms $\partial_n \colon C_n \to C_{n-1}$, which we call **boundary operators**. That is,

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \equiv 0,$$

where 0 denotes the trivial group and $C_i \equiv 0$ for $i \leq 0$. We also require the composition of any two consecutive boundary operators to be zero. That is, for all n,

$$\partial_n \circ \partial_{n+1} = 0,$$

i.e., the constant map to the group identity in C_{n-1}. This means $\mathrm{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$.

Now since each C_n is abelian, $\mathrm{im}(\partial_{n+1})$ is a normal subgroup of $\ker(\partial_n)$. And we want to mod out by this subgroup, i.e., consider everything in $\mathrm{im}(\partial_{n+1})$ equivalent and partition $\ker(\partial_n)$ using this equivalence relation.

We define the ***n*-th homology group of *X*** to be the factor group (or quotient module)

$$H_n(X) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1}).$$

We also use the notation $\ker(\partial_n) = Z_n(X)$ and $\mathrm{im}(\partial_{n+1}) = B_n(X)$, so

$$H_n(X) = Z_n(X)/B_n(X).$$

Computing these two groups is usually rather difficult since they are very large groups. On the other hand, we do have tools which make the task easier.

The *simplicial homology* groups $H_n(X)$ of a *simplicial complex* X are defined using the simplicial chain complex $C(X)$, with $C(X)_n$ the free abelian group generated by the n -simplices of X . The *singular homology* groups $H_n(X)$ are defined for any topological space X , and agree with the simplicial homology groups for a simplicial complex.

A chain complex is said to be exact if the image of the (n + 1)-th map is always equal to the kernel of the nth map. The homology groups of X therefore measure "how far" the chain complex associated to X is from being exact.

Cohomology groups are formally similar: one starts with a cochain complex, which is the same as a chain complex but whose arrows, now denoted d^n point in the direction of increasing n rather than decreasing n; then the groups $\ker(d^n) = Z^n(X)$ and $\mathrm{im}(d^{n-1}) = B^n(X)$ follow from the same description and

$$H^n(X) = Z^n(X)/B^n(X), \text{ as before.}$$

Examples

The motivating example comes from algebraic topology: the **simplicial homology** of a simplicial complex X . Here A_n is the free abelian group or module whose generators are the n-dimensional oriented simplexes of X . The mappings are called the *boundary mappings* and send the simplex with vertices

$$(a[0], a[1], \dots, a[n])$$

to the sum

$$\sum_{i=0}^{n}(-1)^i(a[0], \dots, a[i-1], a[i+1], \dots, a[n])$$

(which is considered 0 if $n = 0$).

If we take the modules to be over a field, then the dimension of the n-th homology of X turns out to be the number of "holes" in X at dimension n.

Using this example as a model, one can define a singular homology for any topological space X . We define a chain complex for X by taking A_n to be the free abelian group (or free module) whose generators are all continuous maps from n-dimensional simplices into X . The homomorphisms ∂_n arise from the boundary maps of simplices.

In abstract algebra, one uses homology to define derived functors, for example the Tor functors. Here one starts with some covariant additive functor F and some module X . The chain complex for X is defined as follows: first find a free module F_1 and a surjective homomorphism $p_1 : F_1 \to X$. Then one finds a free module F_2 and a surjective homomorphism $p_2 : F_2 \to \ker(p_1)$. Continuing in this fashion, a sequence of free modules F_n and homomorphisms p_n can be defined. By applying the functor F to this sequence, one obtains a chain complex; the homology H_n of this complex depends only on F and X and is, by definition, the n-th derived functor of F, applied to X .

Homology functors

Chain complexes form a category: A morphism from the chain complex $(d_n\colon A_n \to A_{n-1})$ to the chain complex $(e_n\colon B_n \to B_{n-1})$ is a sequence of homomorphisms $f_n\colon A_n \to B_n$ such that $f_{n-1} \circ d_n = e_n \circ f_n$ for all n.

The n-th homology H_n can be viewed as a covariant functor from the category of chain complexes to the category of abelian groups (or modules).

If the chain complex depends on the object X in a covariant manner (meaning that any morphism $X \to Y$ induces a morphism from the chain complex of X to the chain complex of Y), then the H_n are covariant functors from the category that X belongs to into the category of abelian groups (or modules).

The only difference between homology and cohomology is that in cohomology the chain complexes depend in a *contravariant* manner on X, and that therefore the homology groups (which are called *cohomology groups* in this context and denoted by H^n) form *contravariant* functors from the category that X belongs to into the category of abelian groups or modules.

Properties

If $(d_n : A_n \to A_{n-1})$ is a chain complex such that all but finitely many A_n are zero, and the others are finitely generated abelian groups (or finite dimensional vector spaces), then we can define the *Euler characteristic*

$$\chi = \sum (-1)^n \operatorname{rank}(A_n)$$

(using the rank in the case of abelian groups and the Hamel dimension in the case of vector spaces). It turns out that the Euler characteristic can also be computed on the level of homology:

$$\chi = \sum (-1)^n \operatorname{rank}(H_n)$$

and, especially in algebraic topology, this provides two ways to compute the important invariant χ for the object X which gave rise to the chain complex.

Every short exact sequence

$$0 \to A \to B \to C \to 0$$

of chain complexes gives rise to a long exact sequence of homology groups

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to H_{n-2}(A) \to \cdots.$$

All maps in this long exact sequence are induced by the maps between the chain complexes, except for the maps $H_n(C) \to H_{n-1}(A)$. The latter are called *connecting homomorphisms* and are provided by the snake lemma.

History

The homology group was developed by Emmy Noether[1] [2] and, independently, by Leopold Vietoris and Walther Mayer, in the period 1925–28.[3] Prior to this, topological classes in combinatorial topology were not formally considered as abelian groups. The spread of homology groups marked the change of terminology and viewpoint from "combinatorial topology" to "algebraic topology".[4]

See also

- Simplicial homology
- Singular homology
- Cellular homology
- Homology theory
- Homological algebra
- Cohomology

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[1] Hilton 1988, p. 284

[2] For example *L'émergence de la notion de group d'homologie*, Nicolas Basbois (PDF) (http://math1.unice.fr/~abbrugia/SPF/publi/expo13.pdf), in French, note 41, explicitly names Noether as inventing the homology group.

[3] Hirzebruch, Friedrich, "Emmy Noether and Topology" in Teicher 1999, p. 61–63.

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Homology theory

In mathematics, **homology theory** is the axiomatic study of the intuitive geometric idea of *homology of cycles* on topological spaces. It can be broadly defined as the study of homology theories on topological spaces.

Simple explanation

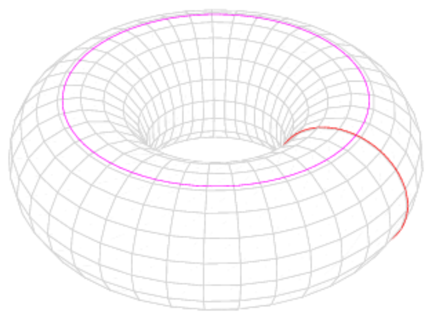
At the intuitive level *homology* is taken to be an equivalence relation, such that chains *C* and *D* are *homologous* on the space *X* if the chain *C* − *D* is a *boundary* of a chain of one dimension higher. The simplest case is in graph theory, with *C* and *D* vertices and homology with a meaning coming from the oriented edge *E* from *P* to *Q* having boundary *Q* — *P*. A collection of edges from *D* to *C*, each one joining up to the one before, is a homology. In general, a *k*-chain is thought of as a formal combination

$$\sum a_i d_i$$

where the a_i are integers and the d_i are *k*-dimensional simplices on *X*. The boundary concept here is that taken over from the boundary of a simplex; it allows a high-dimensional concept which for *k* = 1 is the kind of telescopic cancellation seen in the graph theory case. This explanation is in the style of 1900, and proved somewhat naive, technically speaking.

Example of a torus surface

For example if *X* is a 2-torus *T*, a one-dimensional *cycle* on *T* is in intuitive terms a linear combination of curves drawn on *T*, which closes up on itself (cycle condition, equivalent to having no net boundary). If *C* and *D* are cycles each wrapping once round *T* in the same way, we can find explicitly an oriented area on *T* with boundary *C* − *D*. One can prove that the homology classes of 1-cycles with integer coefficients form a free abelian group with two generators, one generator for each of the two different ways round the 'doughnut'.


A torus with generators colored in pink and red.

The nineteenth century

This level of understanding was common property in the mathematics of the nineteenth century, starting with the idea of Riemann surface. At the end of the century, the work of Poincaré had provided a much more general, though still intuitively-based, setting.

For example, it is considered that the general Stokes' theorem was first stated in 1899 by Poincaré: it involves necessarily both an *integrand* (we would now say, a differential form), and a region of integration (a *p*-chain), with two kinds of *boundary* operators, one of which in modern terms is the exterior derivative, and the other a geometric boundary operator on chains that includes *orientation* and can be used for homology theory. The two *boundaries* appear as adjoint operators, with respect to integration.

Twentieth century beginnings

Rather loose, geometric arguments with homology were only gradually replaced at the beginning of the twentieth century by rigorous techniques. To begin with, the style of the era was to use combinatorial topology (the fore-runner of today's algebraic topology). That assumes that the spaces treated are simplicial complexes, while the most interesting spaces are usually manifolds, so that artificial triangulations have to be introduced to apply the tools. Pioneers such as Solomon Lefschetz and Marston Morse still preferred a geometric approach. The combinatorial stance did allow Brouwer to prove foundational results such as the simplicial approximation theorem, at the base of the idea that homology is a functor (as it would later be put). Brouwer was able to prove the Jordan curve theorem, basic for complex analysis, and the invariance of domain, using the new tools; and remove the suspicion attached to topological arguments as handwaving.

Towards algebraic topology

The transition to *algebraic* topology is usually attributed to the influence of Emmy Noether, who insisted that homology classes lay in quotient groups — a point of view now so fundamental that it is taken as a definition.[1] In fact Noether in the period from 1920 onwards was with her students elaborating the theory of modules for any ring, giving rise when the two ideas were combined to the concept of *homology with coefficients in a ring*. Before that, coefficients (that is, the sense in which chains are linear combinations of the basic geometric chains traced on the space) had usually been integers, real or complex numbers, or sometimes residue classes mod 2. In the new setting, there would be no reason not to take residues mod 3, for example: to be a cycle is then a more complex geometric condition, exemplified in graph theory terms by having the number of incoming edges at every vertex a multiple of 3. But in algebraic terms, the definitions present no new problem. The universal coefficient theorem explains that homology *with integer coefficients* determines all other homology theories, by use of the tensor product; it is not anodyne, in that (as we would now put it) the tensor product has derived functors that enter into a general formulation.

Cohomology, and singular homology

The 1930s were the decade of the development of cohomology theory, as several research directions grew together and the De Rham cohomology that was implicit in Poincaré's work cited earlier became the subject of definite theorems. Cohomology and homology are *dual* theories, in a sense that required detailed working out; at the same time it was realised that homology, that was, simplicial homology, was far from being at the end of its story. The definition of singular homology avoided the need for the apparatus of triangulations, at a cost of moving to infinitely-generated modules.

Axiomatics and extraordinary theories

The development of algebraic topology from 1940 to 1960 was very rapid, and the role of homology theory was often as 'baseline' theory, easy to compute and in terms of which topologists sought to calculate with other functors. The axiomatisation of homology theory by Eilenberg and Steenrod (the Eilenberg-Steenrod axioms) revealed that what various candidate homology *theories* had in common was, roughly speaking, some exact sequences and in particular the Mayer–Vietoris sequence, and the *dimension axiom* that calculated the homology of the point. The dimension axiom was relaxed to admit the (co)homology derived from topological K-theory, and cobordism theory, in a vast extension to the **extraordinary (co)homology theories** that became standard in homotopy theory. These can be easily characterised for the category of CW complexes.

- List of cohomology theories

Current state of homology theory

For more general (i.e. less well-behaved) spaces, recourse to ideas from sheaf theory brought some extension of homology theories, particularly the Borel-Moore homology for locally compact spaces.

The basic chain complex apparatus of homology theory has long since become a separate piece of technique in homological algebra, and has been applied independently, for example to group cohomology. Therefore there is no longer one homology theory, but many homology and cohomology theories in mathematics.

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[1] Hilton 1988, p. 284

Homological algebra

Homological algebra is the branch of mathematics which studies homology in a general algebraic setting. It is a relatively young discipline, whose origins can be traced to investigations in combinatorial topology (a precursor to algebraic topology) and abstract algebra (theory of modules and syzygies) at the end of the 19th century, chiefly by Henri Poincaré and David Hilbert.

The development of homological algebra was closely intertwined with the emergence of category theory. By and large, homological algebra is the study of homological functors and the intricate algebraic structures that they entail. The hidden fabric of mathematics is woven of **chain complexes**, which manifest themselves through their homology and cohomology. Homological algebra affords the means to extract information contained in these complexes and present it in the form of homological invariants of rings, modules, topological spaces, and other 'tangible' mathematical objects. A powerful tool for doing this is provided by spectral sequences.

From its very origins, homological algebra has played an enormous role in algebraic topology. Its sphere of influence has gradually expanded and presently includes commutative algebra, algebraic geometry, algebraic number theory, representation theory, mathematical physics, operator algebras, complex analysis, and the theory of partial differential equations. K-theory is an independent discipline which draws upon methods of homological algebra, as does the noncommutative geometry of Alain Connes.

Chain complexes and homology

The **chain complex** is the central notion of homological algebra. It is a sequence $(C_\bullet, d_\bullet)$ of abelian groups and group homomorphisms, with the property that the composition of any two consecutive maps is zero:

$$C_\bullet : \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots, \quad d_n \circ d_{n+1} = 0.$$

The elements of C_n are called *n*-**chains** and the homomorphisms d_n are called the **boundary maps** or **differentials**. The **chain groups** C_n may be endowed with extra structure; for example, they may be vector spaces or modules over a fixed ring *R*. The differentials must preserve the extra structure if it exists; for example, they must be linear maps or homomorphisms of *R*-modules. For notational convenience, restrict attention to abelian groups (more correctly, to the category **Ab** of abelian groups); a celebrated theorem by Barry Mitchell implies the results will generalize to any abelian category. Every chain complex defines two further sequences of abelian groups, the **cycles** Z_n = Ker d_n and the **boundaries** B_n = Im d_{n+1}, where Ker *d* and Im *d* denote the kernel and the image of *d*. Since the composition of two consecutive boundary maps is zero, these groups are embedded into each other as

$$B_n \subseteq Z_n \subseteq C_n.$$

Subgroups of abelian groups are automatically normal; therefore we can define the *n*th **homology group** $H_n(C)$ as the factor group of the *n*-cycles by the *n*-boundaries,

$$H_n(C) = Z_n/B_n = \operatorname{Ker} d_n/\operatorname{Im} d_{n+1}.$$

A chain complex is called **acyclic** or an **exact sequence** if all its homology groups are zero.

Chain complexes arise in abundance in algebra and algebraic topology. For example, if *X* is a topological space then the singular chains $C_n(X)$ are formal linear combinations of continuous maps from the standard *n*-simplex into *X*; if *K* is a simplicial complex then the simplicial chains $C_n(K)$ are formal linear combinations of the *n*-simplices of *X*; if $A = F/R$ is a presentation of an abelian group *A* by generators and relations, where *F* is a free abelian group spanned by the generators and *R* is the subgroup of relations, then letting $C_1(A) = R$, $C_0(A) = F$, and $C_n(A) = 0$ for all other *n* defines a sequence of abelian groups. In all these cases, there are natural differentials d_n making C_n into a chain complex, whose homology reflects the structure of the topological space *X*, the simplicial complex *K*, or the abelian group *A*. In the case of topological spaces, we arrive at the notion of singular homology, which plays a fundamental role in investigating the properties of such spaces, for example, manifolds.

On a philosophical level, homological algebra teaches us that certain chain complexes associated with algebraic or geometric objects (topological spaces, simplicial complexes, *R*-modules) contain a lot of valuable algebraic information about them, with the homology being only the most readily available part. On a technical level, homological algebra provides the tools for manipulating complexes and extracting this information. Here are two general illustrations.

- Two objects *X* and *Y* are connected by a map *f* between them. Homological algebra studies the relation, induced by the map *f*, between chain complexes associated to *X* and *Y* and their homology. This is generalized to the case of several objects and maps connecting them. Phrased in the language of category theory, homological algebra studies the functorial properties of various constructions of chain complexes and of the homology of these complexes.
- An object *X* admits multiple descriptions (for example, as a topological space and as a simplicial complex) or the complex $C_\bullet(X)$ is constructed using some 'presentation' of *X*, which involves non-canonical choices. It is important to know the effect of change in the description of *X* on chain complexes associated to *X*. Typically, the complex and its homology $H_\bullet(C)$ are functorial with respect to the presentation; and the homology (although not the complex itself) is actually independent of the presentation chosen, thus it is an invariant of *X*.

Functoriality

A continuous map of topological spaces gives rise to a homomorphism between their *n*th homology groups for all *n*. This basic fact of algebraic topology finds a natural explanation through certain properties of chain complexes. Since it is very common to study several topological spaces simultaneously, in homological algebra one is led to simultaneous consideration of multiple chain complexes.

A **morphism** between two chain complexes, $F: C_\bullet \to D_\bullet$, is a family of homomorphisms of abelian groups $F_n: C_n \to D_n$ that commute with the differentials, in the sense that $F_{n-1} \bullet d_n^C = d_n^D \bullet F_n$ for all *n*. A morphism of chain complexes induces a morphism $H_\bullet(F)$ of their homology groups, consisting of the homomorphisms $H_n(F)\colon H_n(C) \to H_n(D)$ for all *n*. A morphism *F* is called a **quasi-isomorphism** if it induces an isomorphism on the *n*th homology for all *n*.

Many constructions of chain complexes arising in algebra and geometry, including singular homology, have the following functoriality property: if two objects *X* and *Y* are connected by a map *f*, then the associated chain complexes are connected by a morphism $F = C(f)$ from $C_\bullet(X)$ to $C_\bullet(Y)$, and moreover, the composition $g \bullet f$ of maps $f\colon X \to Y$ and $g\colon Y \to Z$ induces the morphism $C(g \bullet f)$ from $C_\bullet(X)$ to $C_\bullet(Z)$ that coincides with the composition $C(g) \cdot C(f)$. It follows that the homology groups $H_\bullet(C)$ are functorial as well, so that morphisms between algebraic or topological objects give rise to compatible maps between their homology.

The following definition arises from a typical situation in algebra and topology. A triple consisting of three chain complexes $L_\bullet, M_\bullet, N_\bullet$ and two morphisms between them, $f : L_\bullet \to M_\bullet, g : M_\bullet \to N_\bullet$, is called an **exact triple**, or a **short exact sequence of complexes**, and written as

$$0 \longrightarrow L_\bullet \xrightarrow{f} M_\bullet \xrightarrow{g} N_\bullet \longrightarrow 0,$$

if for any n, the sequence

$$0 \longrightarrow L_n \xrightarrow{f_n} M_n \xrightarrow{g_n} N_n \longrightarrow 0$$

is a short exact sequence of abelian groups. By definition, this means that f_n is an injection, g_n is a surjection, and Im f_n = Ker g_n. One of the most basic theorems of homological algebra, sometimes known as the zig-zag lemma, states that, in this case, there is a **long exact sequence in homology**

$$\ldots \longrightarrow H_n(L) \xrightarrow{H_n(f)} H_n(M) \xrightarrow{H_n(g)} H_n(N) \xrightarrow{\delta_n} H_{n-1}(L) \xrightarrow{H_{n-1}(f)} H_{n-1}(M) \longrightarrow \ldots,$$

where the homology groups of *L*, *M*, and *N* cyclically follow each other, and δ_n are certain homomorphisms determined by *f* and *g*, called the **connecting homomorphisms**. Topological manifestations of this theorem include the Mayer–Vietoris sequence and the long exact sequence for relative homology.

Foundational aspects

Cohomology theories have been defined for many different objects such as topological spaces, sheaves, groups, rings, Lie algebras, and C*-algebras. The study of modern algebraic geometry would be almost unthinkable without sheaf cohomology.

Central to homological algebra is the notion of exact sequence; these can be used to perform actual calculations. A classical tool of homological algebra is that of derived functor; the most basic examples are functors Ext and Tor.

With a diverse set of applications in mind, it was natural to try to put the whole subject on a uniform basis. There were several attempts before the subject settled down. An approximate history can be stated as follows:

- Cartan-Eilenberg: In their 1956 book "Homological Algebra", these authors used projective and injective module resolutions.
- 'Tohoku': The approach in a celebrated paper by Alexander Grothendieck which appeared in the Second Series of the Tohoku Mathematical Journal in 1957, using the abelian category concept (to include sheaves of abelian groups).
- The derived category of Grothendieck and Verdier. Derived categories date back to Verdier's 1967 thesis. They are examples of triangulated categories used in a number of modern theories.

These move from computability to generality.

The computational sledgehammer *par excellence* is the spectral sequence; these are essential in the Cartan-Eilenberg and Tohoku approaches where they are needed, for instance, to compute the derived functors of a composition of two functors. Spectral sequences are less essential in the derived category approach, but still play a role whenever concrete computations are necessary.

There have been attempts at 'non-commutative' theories which extend first cohomology as *torsors* (important in Galois cohomology).

See also

- Homotopical algebra

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Group cohomology

In abstract algebra, homological algebra, algebraic topology and algebraic number theory, as well as in applications to group theory proper, **group cohomology** is a way to study groups using a sequence of functors H^n. The study of fixed points of groups acting on modules and quotient modules is a motivation, but the cohomology can be defined using various constructions. There is a dual theory, group homology, and a generalization to non-abelian coefficients. These algebraic ideas are closely related to topological ideas. A great deal is known about the cohomology of groups, including interpretations of low dimensional cohomology, functorality, and how to change groups. The history of group cohomology began in the 1920s, matured in the late 1940s, and continues as an area of active research today.

Motivation

A general paradigm in group theory is that a group G should be studied via its group representations. A slight generalization of those representations are the G-modules: a G-module is an abelian group M together with a group action of G on M, with every element of G acting as an endomorphism of M. In the sequel we will write G multiplicatively and M additively.

Given such a G-module M, it is natural to consider the subgroup of G-invariant elements:

$$M^G = \{x \in M : \forall g \in G \; gx = x\}.$$

Now, if N is a submodule of M (i.e. a subgroup of M mapped to itself by the action of G), it isn't in general true that the invariants in M/N are found as the quotient of the invariants in M by N: being invariant 'up to something in N ' is broader. The first group cohomology $H^1(G,N)$ precisely measures the difference. The group cohomology functors H^n in general measure the extent to which taking invariants doesn't respect exact sequences. This is expressed by a long exact sequence.

Formal constructions

In this article, G is a finite group. The collection of all G-modules is a category (the morphisms are group homomorphisms f with the property $f(gx) = g(f(x))$ for all g in G and x in M). This category of G-modules is an abelian category with enough injectives (since it is isomorphic to the category of all modules over the group ring $\mathbb{Z}[G]$).

Sending each module M to the group of invariants M^G yields a functor from this category to the category $\mathfrak{Ab}$ of abelian groups. This functor is left exact but not necessarily right exact. We may therefore form its right derived functors; their values are abelian groups and they are denoted by $H^n(G,M)$, "the n-th cohomology group of G with coefficients in M". $H^0(G,M)$ is identified with M^G.

Long exact sequence of cohomology

In practice, one often computes the cohomology groups using the following fact: if

$$0 \to L \to M \to N \to 0$$

is a short exact sequence of G-modules, then a long exact sequence

$$0 \to L^G \to M^G \to N^G \xrightarrow{\delta^0} H^1(G,L) \to H^1(G,M) \to H^1(G,N) \xrightarrow{\delta^1} H^2(G,L) \to \cdots$$

is induced. The maps δ^n are called the "connecting homomorphisms" and can be obtained from the snake lemma.[1]

Cochain complexes

Rather than using the machinery of derived functors, the cohomology groups can also be defined more concretely, as follows.[2] For $n \geq 0$, let $C^n(G, M)$ be the group of all functions from G^n to M. This is an abelian group; its elements are called the (inhomogeneous) ***n*-cochains**. The **coboundary homomorphisms**

$$d^n : C^n(G,M) \to C^{n+1}(G,M)$$

are defined as

$$\begin{aligned}(d^n\varphi)(g_1,\dots,g_{n+1}) &= g_1 \cdot \varphi(g_2,\dots,g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i \varphi(g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_{n+1}) \\ &+ (-1)^{n+1}\varphi(g_1,\dots,g_n)\end{aligned}$$

The crucial thing to check here is

$$d^{n+1} \circ d^n = 0$$

thus we have a cochain complex and we can compute cohomology. For $n \geq 0$, define the group of ***n*-cocycles** as:

$$Z^n(G,M) = \ker(d^n)$$

and the group of n **-coboundaries** as

$$\begin{cases} B^0(G,M) = 0 \\ B^n(G,M) = \operatorname{im}(d^{n-1}) \ n \geq 1 \end{cases}$$

and

$$H^n(G,M) = Z^n(G,M)/B^n(G,M).$$

The functors Ext^n and formal definition of group cohomology

Yet another approach is to treat *G*-modules as modules over the group ring ℤ[*G*], which allows one to define group cohomology via Ext functors:

$$H^n(G,M)=\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},M),$$

where *M* is a ℤ[*G*]-module.

Here ℤ is treated as the trivial *G*-module: every element of *G* acts as the identity. These Ext groups can also be computed via a projective resolution of ℤ, the advantage being that such a resolution only depends on *G* and not on *M*. We recall the definition of Ext more explicitly for this context. Let *F* be a projective ℤ[*G*]-resolution (e.g. a free ℤ[*G*]-resolution) of the trivial ℤ[*G*]-module ℤ:

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z}.$$

e.g., one may always take the resolution of group rings, $F_n=\mathbb{Z}[G^{n+1}]$, with morphisms

$$f_n:\mathbb{Z}[G^{n+1}]\to\mathbb{Z}[G^n],\quad (g_0,g_1,\ldots,g_n)\mapsto\sum_{i=0}^{n}(-1)^i(g_0,\ldots,\widehat{g_i},\ldots,g_n).$$

Recall that for ℤ[*G*]-modules *N* and *M*, $Hom_G(N, M)$ is an abelian group consisting of ℤ[*G*]-homomorphisms from *N* to *M*. Since $Hom_G(–, M)$ is a contravariant functor and reverses the arrows, applying $Hom_G(–, M)$ to *F* termwise produces a cochain complex $Hom_G(F, M)$:

$$\ldots \leftarrow \mathrm{Hom}_G(F_n,M)\leftarrow \mathrm{Hom}_G(F_{n-1},M)\leftarrow\cdots\leftarrow \mathrm{Hom}_G(F_0,M)\leftarrow \mathrm{Hom}_G(\mathbb{Z},M).$$

The cohomology groups $H^*(G,M)$ of *G* with coefficients in *M* are defined as the cohomology of the above cochain complex:

$$H^n(G,M)=H^n(Hom_G(F, M))$$

for $n \geq 0$.

Group homology

Dually to the construction of group cohomology there is the following definition of **group homology**: given a *G*-module *M*, set *DM* to be the submodule generated by elements of the form $g{\cdot}m-m$, $g \in G$, $m \in M$. Assigning to *M* its so-called *co-invariants*, the quotient

$$M_G := M/DM,$$

is a right exact functor. Its left derived functors are by definition the group homology

$$H_n(G, M).$$

The covariant functor which assigns M_G to *M* is isomorphic to the functor which sends *M* to $\mathbb{Z}\otimes_{\mathbb{Z}[G]}M$, where $\mathbb{Z}$ is endowed with the trivial *G*-action. Hence one also gets an expression for group homology in terms of the Tor functors,

$$H_n(G,M)=\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},M)$$

Recall that the tensor product $N\otimes_{\mathbb{Z}[G]}M$ is defined whenever *N* is a right ℤ[*G*]-module and *M* is a left ℤ[*G*]-module. If *N* is a left ℤ[*G*]-module, we turn it into a right ℤ[*G*]-module by setting $a\,g = g^{-1}a$ for every $g \in G$ and every $a \in N$. This convention allows to define the tensor product $N\otimes_{\mathbb{Z}[G]}M$ in the case where both *M* and *N* are left ℤ[*G*]-modules.

Specifically, the homology groups $H_n(G, M)$ can be computed as follows. Start with a projective resolution *F* of the trivial ℤ[*G*]-module ℤ, as in the previous section. Apply the covariant functor $\otimes_{\mathbb{Z}[G]}M$ to *F* termwise to get a chain complex $F\otimes_{\mathbb{Z}[G]}M$:

$$\cdots\to F_n\otimes_{\mathbb{Z}[G]}M\to F_{n-1}\otimes_{\mathbb{Z}[G]}M\to\cdots\to F_0\otimes_{\mathbb{Z}[G]}M\to\mathbb{Z}\otimes_{\mathbb{Z}[G]}M.$$

Then $H_n(G, M)$ are the homology groups of this chain complex, $H_n(G,M)=H_n(F\otimes_{\mathbb{Z}[G]}M)$ for $n \geq 0$.

Group homology and cohomology can be treated uniformly for some groups, especially finite groups, in terms of complete resolutions and the Tate cohomology groups.

Functorial maps in terms of cochains

Connecting homomorphisms

For a short exact sequence $0 \to L \to M \to N \to 0$, the connecting homomorphisms $\delta^n : H^n(G, N) \to H^{n+1}(G, L)$ can be described in terms of inhomogeneous cochains as follows.[3] If c is an element of $H^n(G, N)$ represented by an n-cocycle $\varphi : G^n \to N$, then $\delta^n(c)$ is represented by $d^n(\psi)$, where ψ is an n-cochain $G^n \to M$ "lifting" φ (i.e. such that φ is the composition of ψ with the surjective map $M \to N$).

Non-abelian group cohomology

Using the G-invariants and the 1-cochains, one can construct the zeroth and first group cohomology for a group G with coefficients in a non-abelian group. Specifically, a G-group is a (not necessarily abelian) group A together with an action by G.

The *zeroth cohomology of G with coefficients in* A *is*

$$H^0(G, A) = A^G,$$

which is a subgroup of A.

The *first cohomology of G with coefficents in A* is defined as above using 1-cocycles and 1-coboundaries. However, it is generally not a group when A is non-abelian. It instead has the structure of a pointed set.

Using explicit calculations, one still obtains a *truncated* long exact sequence in cohomology. Specifically, let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of G-groups, then there is an exact sequence of pointed sets

$$0 \to A^G \to B^G \to C^G \to H^1(G, A) \to H^1(G, B) \to H^1(G, C).$$

Connections with topological cohomology theories

Group cohomology can be related to topological cohomology theories: to the topological group G there is an associated classifying space BG. (If G has no topology about which we care, then we assign the discrete topology to G. In this case, BG is an Eilenberg-MacLane space $K(G,1)$, whose fundamental group is G and whose higher homotopy groups vanish). The n-th cohomology of BG, with coefficients in M (in the topological sense), is the same as the group cohomology of G with coefficients in M. This will involve a local coefficient system unless M is a trivial G-module. The connection holds because the total space EG is contractible, so its chain complex forms a projective resolution of M. These connections are explained in (Adem-Milgram 2004), Chapter II.

When M is a ring with trivial G-action, we inherit good properties which are familiar from the topological context: in particular, there is a cup product under which

$$H^*(G; M) = \bigoplus_n H^n(G; M)$$

is a graded module, and a Künneth formula applies.

If, furthermore, $M=k$ is a field, then $H^*(G;k)$ is a graded k-algebra. In this case, the Künneth formula yields

$$H^*(G_1 \times G_2; k) \cong H^*(G_1; k) \otimes H^*(G_2; k).$$

For example, let G be the group with two elements, under the discrete topology. The real projective space $\mathbb{R}P^{\infty}$ is a classifying space for G. Let $k=\mathbf{F}_2$, the field of two elements. Then

$$H^*(G; k) \cong k[x],$$

a polynomial k-algebra on a single generator, since this is the cellular cohomology ring of □P^{∞}.

Hence, as a second example, if G is an elementary abelian 2-group of rank r, and $k=\mathbf{F}_2$, then the Künneth formula gives

$$H^*(G;k) \cong k[x_1, \ldots, x_r],$$

a polynomial k-algebra generated by r classes in $H^1(G;k)$.

Properties

In the following, let M be a G-module.

Functoriality

Group cohomology depends contravariantly on the group G, in the following sense: if $f : H \to G$ is a group homomorphism, then we have a naturally induced morphism $H^n(G,M) \to H^n(H,M)$ (where in the latter case, M is treated as an H-module via f).

Given a morphism of G-modules $M \to N$, one gets a morphism of cohomology groups in the $H^n(G,M) \to H^n(G,N)$.

H^1

The first cohomology group is the quotient of the so-called *crossed homomorphisms*, i.e. maps (of sets) $f : G \to M$ satisfying $f(ab) = f(a) + af(b)$ for all a, b in G, modulo the so-called *principal crossed homomorphisms*, i.e. maps $f : G \to M$ given by $f(a) = am - m$ for some fixed $m \in M$. This follows from the definition of cochains above.

If the action of G on M is trivial, then the above boils down to $H^1(G, M) = Hom(G, M)$, the group of group homomorphisms $G \to M$.

H^2

If M is a trivial G-module (i.e. the action of G on M is trivial), the second cohomology group $H^2(G,M)$ is in one-to-one correspondence with the set of central extensions of G by M (up to a natural equivalence relation). More generally, if the action of G on M is nontrivial, $H^2(G,M)$ classifies the isomorphism classes of all extensions of G by M in which the induced action of G on M by inner automorphisms agrees with the given action.

Change of group

The Hochschild-Serre spectral sequence relates the cohomology of a normal subgroup N of G and the quotient G/N to the cohomology of the group G (for (pro-)finite groups G).

History and relation to other fields

The low dimensional cohomology of a group was classically studied in other guises, long before the notion of group cohomology was formulated in 1943-45. The first theorem of the subject can be identified as Hilbert's Theorem 90 in 1897; this was recast into *Noether's equations* in Galois theory (an appearance of cocycles for H^1). The idea of *factor sets* for the extension problem for groups (connected with H^2) arose in the work of Hölder (1893), in Issai Schur's 1904 study of projective representations, in Schreier's 1926 treatment, and in Richard Brauer's 1928 study of simple algebras and the Brauer group. A fuller discussion of this history may be found in (Weibel 1999, pp. 806-811).

In 1941, while studying $H_2(G, \mathbb{Z})$ (which plays a special role in groups), Hopf discovered what is now called **Hopf's integral homology formula** (Hopf 1942), which is identical to Schur's formula for the Schur multiplier of a finite, finitely presented group:

$$H_2(G, \mathbb{Z}) \cong (R \cap [F, F])/[F, R],$$ where $G \cong F/R$ and F is a free group.

Hopf's result led to the independent discovery of group cohomology by several groups in 1943-45: Eilenberg and Mac Lane in the USA (Rotman 1995, p. 358); Hopf and Eckmann in Switzerland; and Freudenthal in the Netherlands (Weibel 1999, p. 807). The situation was chaotic because communication between these countries was difficult during World War II.

From a topological point of view, the homology and cohomology of G was first defined as the homology and cohomology of a model for the topological classifying space *BG* as discussed in #Connections with topological cohomology theories above. In practice, this meant using topology to produce the chain complexes used in formal algebraic definitions. From a module-theoretic point of view this was integrated into the Cartan-Eilenberg theory of Homological algebra in the early 1950s.

The application in algebraic number theory to class field theory provided theorems valid for general Galois extensions (not just abelian extensions). The cohomological part of class field theory was axiomatized as the theory of class formations. In turn, this led to the notion of Galois cohomology and étale cohomology (which builds on it) (Weibel 1999, p. 822). Some refinements in the theory post-1960 have been made, such as continuous cocycles and Tate's redefinition, but the basic outlines remain the same. This is a large field, and now basic in the theories of algebraic groups.

The analogous theory for Lie algebras, called Lie algebra cohomology, was first developed in the late 1940s, by Chevalley-Eilenberg, and Koszul (Weibel 1999, p. 810). It is formally similar, using the corresponding definition of *invariant* for the action of a Lie algebra. It is much applied in representation theory, and is closely connected with the BRST quantization of theoretical physics.

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Cohomology

In mathematics, specifically in algebraic topology, **cohomology** is a general term for a sequence of abelian groups defined from a cochain complex. That is, cohomology is defined as the abstract study of **cochains**, cocycles, and coboundaries. Cohomology can be viewed as a method of assigning algebraic invariants to a topological space that has a more refined algebraic structure than does homology. Cohomology arises from the algebraic dualization of the construction of homology. In less abstract language, cochains in the fundamental sense should assign 'quantities' to the *chains* of homology theory.

From its beginning in topology, this idea became a dominant method in the mathematics of the second half of the twentieth century; from the initial idea of *homology* as a topologically invariant relation on *chains*, the range of applications of homology and cohomology theories has spread out over geometry and abstract algebra. The terminology tends to mask the fact that in many applications *cohomology*, a contravariant theory, is more natural than *homology*. At a basic level this has to do with functions and pullbacks in geometric situations: given spaces *X* and *Y*, and some kind of function *F* on *Y*, for any mapping ƒ: $X \to Y$ composition with ƒ gives rise to a function *F* o ƒ on *X*. Cohomology groups often also have a natural product, the cup product, which gives them a ring structure. Because of this feature, cohomology is a stronger invariant than homology, as it can differentiate between certain algebraic objects that homology cannot.

Definition

For a topological space *X*, the **cohomology group** $H^n(X;G)$, with coefficents in *G*, is defined to be the quotient Ker(δ)/Im(δ) at $C^n(X;G)$ in the cochain complex

$$\ldots \leftarrow C^{n+1}(X;G) \xleftarrow{\delta} C^n(X;G) \leftarrow \ldots \leftarrow C^0(X;G) \leftarrow 0.$$

Elements in Ker(δ) are **cocycles** and elements in Im(δ) are **coboundaries**.

History

Although cohomology is fundamental to modern algebraic topology, its importance was not seen for some 40 years after the development of homology. The concept of *dual cell structure*, which Henri Poincaré used in his proof of his Poincaré duality theorem, contained the germ of the idea of cohomology, but this was not seen until later.

There were various precursors to cohomology. In the mid-1920s, J.W. Alexander and Solomon Lefschetz founded the intersection theory of cycles on manifolds. On an *n*-dimensional manifold *M*, a *p*-cycle and a *q*-cycle with nonempty intersection will, if in general position, have intersection a $(p+q-n)$-cycle. This enables us to define a multiplication of homology classes

$$H_p(M) \times H_q(M) \to H_{p+q-n}(M).$$

Alexander had by 1930 defined a first cochain notion, based on a p-cochain on a space X having relevance to the small neighborhoods of the diagonal in X^{p+1}.

In 1931, Georges de Rham related homology and exterior differential forms, proving De Rham's theorem. This result is now understood to be more naturally interpreted in terms of cohomology.

In 1934, Lev Pontryagin proved the Pontryagin duality theorem; a result on topological groups. This (in rather special cases) provided an interpretation of Poincaré duality and Alexander duality in terms of group characters.

At a 1935 conference in Moscow, Andrey Kolmogorov and Alexander both introduced cohomology and tried to construct a cohomology product structure.

In 1936 Norman Steenrod published a paper constructing Čech cohomology by dualizing Čech homology.

From 1936 to 1938, Hassler Whitney and Eduard Čech developed the cup product (making cohomology into a graded ring) and cap product, and realized that Poincaré duality can be stated in terms of the cap product. Their theory was still limited to finite cell complexes.

In 1944, Samuel Eilenberg overcame the technical limitations, and gave the modern definition of singular homology and cohomology.

In 1945, Eilenberg and Steenrod stated the axioms defining a homology or cohomology theory. In their 1952 book, *Foundations of Algebraic Topology*, they proved that the existing homology and cohomology theories did indeed satisfy their axioms.[1]

In 1948 Edwin Spanier, building on work of Alexander and Kolmogorov, developed Alexander-Spanier cohomology.

Cohomology theories

Eilenberg-Steenrod theories

A *cohomology theory* is a family of contravariant functors from the category of pairs of topological spaces and continuous functions (or some subcategory thereof such as the category of CW complexes) to the category of Abelian groups and group homomorphisms that satisfies the Eilenberg-Steenrod axioms.

Some cohomology theories in this sense are:

- simplicial cohomology
- singular cohomology
- de Rham cohomology
- Čech cohomology
- sheaf cohomology.

Extraordinary cohomology theories

When one axiom (*dimension axiom*) is relaxed, one obtains the idea of **extraordinary cohomology theory** or **generalized cohomology theory**; this allows theories based on K-theory and cobordism theory. There are others, coming from stable homotopy theory. In this context, singular homology is referred to as **ordinary homology.**

An extraordinary cohomology theory is "determined by its values on a point", in the sense that if one has a space given by contractible spaces (homotopy equivalent to a point), glued together along contractible spaces, as in a simplicial complex, then the cohomology of the space is determined by the cohomology of a point and the combinatorics of the patching, and effectively computable. Formally, this is computed by the excision theorem, or equivalently the Mayer–Vietoris sequence. Thus the cohomology of a point is a fundamental calculation for any extraordinary cohomology theory, though the cohomology of particular spaces is also of interest.

Other cohomology theories

Theories in a broader sense of *cohomology* include:[2]

- André–Quillen cohomology
- BRST cohomology
- Bonar-Claven cohomology
- Bounded cohomology
- Coherent cohomology
- Crystalline cohomology
- Cyclic cohomology
- Deligne cohomology
- Dirac cohomology
- Étale cohomology
- Flat cohomology
- Galois cohomology
- Gel'fand-Fuks cohomology
- Group cohomology
- Harrison cohomology
- Hochschild cohomology
- Intersection cohomology
- Lie algebra cohomology
- Local cohomology
- Motivic cohomology
- Non-abelian cohomology
- Perverse cohomology
- Quantum cohomology
- Schur cohomology
- Spencer cohomology
- Topological André-Quillen cohomology
- Topological Cyclic cohomology
- Topological Hochschild cohomology
- Γ cohomology

See also

- List of cohomology theories

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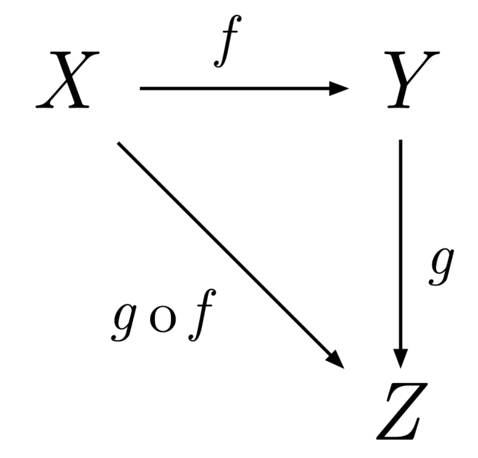
Theory of Categories

Category theory

In mathematics, **category theory** deals in an abstract way with mathematical structures and relationships between them: it abstracts from *sets* and *functions* to *objects* linked in diagrams by *morphisms* or *arrows*.



A category with objects X, Y, Z and morphisms *f*, *g*

One of the simplest examples of a category (which is a very important concept in topology) is that of groupoid, defined as a category whose arrows or morphisms are all invertible. Categories now appear in most branches of mathematics, some areas of theoretical computer science where they correspond to types, and mathematical physics where they can be used to describe vector spaces. Category theory provides both with a unifying notion and terminology. Categories were first introduced by Samuel Eilenberg and Saunders Mac Lane in 1942–45, in connection with algebraic topology.

Category theory has several faces known not just to specialists, but to other mathematicians. A term dating from the 1940s, "general abstract nonsense", refers to its high level of abstraction, compared to more classical branches of mathematics. Homological algebra is category theory in its aspect of organising and suggesting manipulations in abstract algebra. Diagram chasing is a visual method of arguing with abstract "arrows" joined in diagrams. Note that arrows between categories are called functors, subject to specific defining commutativity conditions; moreover, categorical diagrams and sequences can be defined as functors (viz. Mitchell, 1965). An arrow between two functors is a natural transformation when it is subject to certain naturality or commutativity conditions. Both functors and natural transformations are key concepts in category theory, or the "real engines" of category theory. To paraphrase a famous sentence of the mathematicians who founded category theory: 'Categories were introduced to define functors, and functors were introduced to define natural transformations'. Topos theory is a form of abstract sheaf theory, with geometric origins, and leads to ideas such as pointless topology. A topos can also be considered as a specific type of category with two additional topos axioms.

Background

The study of categories is an attempt to *axiomatically* capture what is commonly found in various classes of related *mathematical structures* by relating them to the *structure-preserving functions* between them. A systematic study of category theory then allows us to prove general results about any of these types of mathematical structures from the axioms of a category.

Consider the following example. The class **Grp** of groups consists of all objects having a "group structure". One can proceed to prove theorems about groups by making logical deductions from the set of axioms. For example, it is immediately proved from the axioms that the identity element of a group is unique.

Instead of focusing merely on the individual objects (e.g., groups) possessing a given structure, category theory emphasizes the morphisms – the structure-preserving mappings – *between* these objects; by studying these morphisms, we are able to learn more about the structure of the objects. In the case of groups, the morphisms are the group homomorphisms. A group homomorphism between two groups "preserves the group structure" in a precise sense – it is a "process" taking one group to another, in a way that carries along information about the structure of the first group into the second group. The study of group homomorphisms then provides a tool for studying general properties of groups and consequences of the group axioms.

A similar type of investigation occurs in many mathematical theories, such as the study of continuous maps (morphisms) between topological spaces in topology (the associated category is called **Top**), and the study of smooth functions (morphisms) in manifold theory.

If one axiomatizes relations instead of functions, one obtains the theory of allegories.

Functors

Abstracting again, a category is *itself* a type of mathematical structure, so we can look for "processes" which preserve this structure in some sense; such a process is called a functor. A functor associates to every object of one category an object of another category, and to every morphism in the first category a morphism in the second.

In fact, what we have done is define a category *of categories and functors* – the objects are categories, and the morphisms (between categories) are functors.

By studying categories and functors, we are not just studying a class of mathematical structures and the morphisms between them; we are studying the *relationships between various classes of mathematical structures*. This is a fundamental idea, which first surfaced in algebraic topology. Difficult *topological* questions can be translated into *algebraic* questions which are often easier to solve. Basic constructions, such as the fundamental group or fundamental groupoid [1] of a topological space, can be expressed as fundamental functors [1] to the category of groupoids in this way, and the concept is pervasive in algebra and its applications.

Natural transformation

Abstracting yet again, constructions are often "naturally related" – a vague notion, at first sight. This leads to the clarifying concept of natural transformation, a way to "map" one functor to another. Many important constructions in mathematics can be studied in this context. "Naturality" is a principle, like general covariance in physics, that cuts deeper than is initially apparent.

Historical notes

In 1942–45, Samuel Eilenberg and Saunders Mac Lane introduced categories, functors, and natural transformations as part of their work in topology, especially algebraic topology. Their work was an important part of the transition from intuitive and geometric homology to axiomatic homology theory. Eilenberg and Mac Lane later wrote that their goal was to understand natural transformations; in order to do that, functors had to be defined, which required categories.

Stanislaw Ulam, and some writing on his behalf, have claimed that related ideas were current in the late 1930s in Poland. Eilenberg was Polish, and studied mathematics in Poland in the 1930s. Category theory is also, in some sense, a continuation of the work of Emmy Noether (one of Mac Lane's teachers) in formalizing abstract processes; Noether realized that in order to understand a type of mathematical structure, one needs to understand the processes preserving that structure. In order to achieve this understanding, Eilenberg and Mac Lane proposed an axiomatic formalization of the relation between structures and the processes preserving them.

The subsequent development of category theory was powered first by the computational needs of homological algebra, and later by the axiomatic needs of algebraic geometry, the field most resistant to being grounded in either axiomatic set theory or the Russell-Whitehead view of united foundations. General category theory, an extension of universal algebra having many new features allowing for semantic flexibility and higher-order logic, came later; it is now applied throughout mathematics.

Certain categories called topoi (singular *topos*) can even serve as an alternative to axiomatic set theory as a foundation of mathematics. These foundational applications of category theory have been worked out in fair detail as a basis for, and justification of, constructive mathematics. More recent efforts to introduce undergraduates to categories as a foundation for mathematics include Lawvere and Rosebrugh (2003) and Lawvere and Schanuel (1997).

Categorical logic is now a well-defined field based on type theory for intuitionistic logics, with applications in functional programming and domain theory, where a cartesian closed category is taken as a non-syntactic description of a lambda calculus. At the very least, category theoretic language clarifies what exactly these related areas have in common (in some abstract sense).

Categories, objects, and morphisms

A *category* C consists of the following three mathematical entities:

- A class ob(C), whose elements are called *objects*;
- A class hom(C), whose elements are called morphisms or maps or *arrows*. Each morphism f has a unique *source object* a and *target object* b. We write f: $a \to b$, and we say "f is a morphism from a to b". We write hom(a, b) (or Hom(a, b), or $\hom_C(a, b)$, or Mor(a, b), or $C(a, b)$) to denote the *hom-class* of all morphisms from a to b.
- A binary operation $\circ$, called *composition of morphisms*, such that for any three objects a, b, and c, we have hom(a, b) × hom(b, c) → hom(a, c). The composition of f: $a \to b$ and g: $b \to c$ is written as $g \circ f$ or gf[2], governed by two axioms:
  - Associativity: If $f : a \to b$, $g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
  - Identity: For every object x, there exists a morphism $1_x : x \to x$ called the *identity morphism for x*, such that for every morphism $f : a \to b$, we have $1_b \circ f = f = f \circ 1_a$.

From these axioms, it can be proved that there is exactly one identity morphism for every object. Some authors deviate from the definition just given by identifying each object with its identity morphism.

Relations among morphisms (such as $fg = h$) are often depicted using commutative diagrams, with "points" (corners) representing objects and "arrows" representing morphisms.

Properties of morphisms

Some morphisms have important properties. A morphism $f : a \to b$ is:

- a monomorphism (or *monic*) if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : x \to a$.
- an epimorphism (or *epic*) if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : b \to x$.
- an isomorphism if there exists a morphism $g : b \to a$ with $f \circ g = 1_b$ and $g \circ f = 1_a$.[3]
- an endomorphism if $a = b$. end(a) denotes the class of endomorphisms of a.
- an automorphism if f is both an endomorphism and an isomorphism. aut(a) denotes the class of automorphisms of a.

Functors

Functors are structure-preserving maps between categories. They can be thought of as morphisms in the category of all (small) categories.

A (**covariant**) functor F from a category C to a category D, written F:$C \to D$, consists of:

- for each object x in C, an object $F(x)$ in D; and
- for each morphism $f : x \to y$ in C, a morphism $F(f) : F(x) \to F(y)$,

such that the following two properties hold:

- For every object x in C, $F(1_x) = 1_{F(x)}$;
- For all morphisms $f : x \to y$ and $g : y \to z$, $F(g \circ f) = F(g) \circ F(f)$.

A **contravariant** functor F: $C \to D$, is like a covariant functor, except that it "turns morphisms around" ("reverses all the arrows"). More specifically, every morphism $f : x \to y$ in C must be assigned to a morphism $F(f) : F(y) \to F(x)$ in D. In other words, a contravariant functor is a covariant functor from the opposite category C^{op} to D.

Natural transformations and isomorphisms

A *natural transformation* is a relation between two functors. Functors often describe "natural constructions" and natural transformations then describe "natural homomorphisms" between two such constructions. Sometimes two quite different constructions yield "the same" result; this is expressed by a natural isomorphism between the two functors.

If F and G are (covariant) functors between the categories C and D, then a natural transformation from F to G associates to every object x in C a morphism $\eta_x : F(x) \to G(x)$ in D such that for every morphism $f : x \to y$ in C, we have $\eta_y \circ F(f) = G(f) \circ \eta_x$; this means that the following diagram is commutative:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

The two functors F and G are called *naturally isomorphic* if there exists a natural transformation from F to G such that η_x is an isomorphism for every object x in C.

Universal constructions, limits, and colimits

Using the language of category theory, many areas of mathematical study can be cast into appropriate categories, such as the categories of all sets, groups, topologies, and so on. These categories surely have some objects that are "special" in a certain way, such as the empty set or the product of two topologies, yet in the definition of a category, objects are considered to be atomic, i.e., we *do not know* whether an object A is a set, a topology, or any other abstract concept – hence, the challenge is to define special objects without referring to the internal structure of those objects. But how can we define the empty set without referring to elements, or the product topology without referring to open sets?

The solution is to characterize these objects in terms of their relations to other objects, as given by the morphisms of the respective categories. Thus, the task is to find *universal properties* that uniquely determine the objects of interest.

Indeed, it turns out that numerous important constructions can be described in a purely categorical way. The central concept which is needed for this purpose is called categorical *limit*, and can be dualized to yield the notion of a *colimit*.

Equivalent categories

It is a natural question to ask: under which conditions can two categories be considered to be "essentially the same", in the sense that theorems about one category can readily be transformed into theorems about the other category? The major tool one employs to describe such a situation is called *equivalence of categories*, which is given by appropriate functors between two categories. Categorical equivalence has found numerous applications in mathematics.

Further concepts and results

The definitions of categories and functors provide only the very basics of categorical algebra; additional important topics are listed below. Although there are strong interrelations between all of these topics, the given order can be considered as a guideline for further reading.

- The functor category D^C has as objects the functors from C to D and as morphisms the natural transformations of such functors. The Yoneda lemma is one of the most famous basic results of category theory; it describes representable functors in functor categories.
- Duality: Every statement, theorem, or definition in category theory has a *dual* which is essentially obtained by "reversing all the arrows". If one statement is true in a category C then its dual will be true in the dual category C^{op}. This duality, which is transparent at the level of category theory, is often obscured in applications and can lead to surprising relationships.
- Adjoint functors: A functor can be left (or right) adjoint to another functor that maps in the opposite direction. Such a pair of adjoint functors typically arises from a construction defined by a universal property; this can be seen as a more abstract and powerful view on universal properties.

Higher-dimensional categories

Many of the above concepts, especially equivalence of categories, adjoint functor pairs, and functor categories, can be situated into the context of *higher-dimensional categories*. Briefly, if we consider a morphism between two objects as a "process taking us from one object to another", then higher-dimensional categories allow us to profitably generalize this by considering "higher-dimensional processes".

For example, a (strict) 2-category is a category together with "morphisms between morphisms", i.e., processes which allow us to transform one morphism into another. We can then "compose" these "bimorphisms" both horizontally and vertically, and we require a 2-dimensional "exchange law" to hold, relating the two composition laws. In this context, the standard example is **Cat**, the 2-category of all (small) categories, and in this example, bimorphisms of morphisms are simply natural transformations of morphisms in the usual sense. Another basic example is to consider a 2-category with a single object; these are essentially monoidal categories. Bicategories are a weaker notion of 2-dimensional categories in which the composition of morphisms is not strictly associative, but only associative "up to" an isomorphism.

This process can be extended for all natural numbers n, and these are called n-categories. There is even a notion of *ω-category* corresponding to the ordinal number ω.

Higher-dimensional categories are part of the broader mathematical field of higher-dimensional algebra,a concept introduced by Ronald Brown. For a conversational introduction to these ideas, see John Baez, 'A Tale of n-categories' (1996). [4]

See also

- Domain theory
- Enriched category theory
- Glossary of category theory
- Higher category theory
- Higher-dimensional algebra
- Important publications in category theory
- Timeline of category theory and related mathematics

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[2] Some authors compose in the opposite order, writing *fg* or $f \circ g$ for $g \circ f$. Computer scientists using category theory very commonly write *f;g* for $g \circ f$

[3] Note that a morphism that is both epic and monic is not necessarily an isomorphism! For example, in the category consisting of two objects *A* and *B*, the identity morphisms, and a single morphism *f* from *A* to *B*, *f* is both epic and monic but is not an isomorphism.

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Category (mathematics)

In mathematics, a **category** is an algebraic structure consisting of a collection of "objects", linked together by a collection of "arrows" that have two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. Objects and arrows may be abstract entities of any kind. Categories generalize monoids, groupoids and preorders. In addition, the notion of category provides a fundamental and abstract way to describe mathematical entities and their relationships. This is the central idea of *category theory*, a branch of mathematics which seeks to generalize all of mathematics in terms of objects and arrows, independent of what the objects and arrows represent. Virtually every branch of modern mathematics can be described in terms of categories, and doing so often reveals deep insights and similarities between seemingly different areas of mathematics. For more extensive motivational background and historical notes, see category theory and the list of category theory topics.

Two categories are the same if they have the same collection of objects, the same collection of arrows, and the same associative method of composing any pair of arrows. Two categories may also be considered "equivalent" for purposes of category theory, even if they are not precisely the same. Many well-known categories are conventionally identified by a short capitalized word or abbreviation in bold or italics such as **Set** (category of sets and set functions), **Ring** (category of rings and ring homomorphisms), or **Top** (category of topological spaces and continuous maps).

Definition

A **category** C consists of

- a class ob(C) of **objects**:
- a class hom(C) of **morphisms**, or **arrows**, or **maps**, between the objects. Each morphism f has a unique *source object* a and *target object* b where a and b are in ob(C). We write f: $a \to b$, and we say "f is a morphism from a to b". We write hom(a, b) (or $\hom_C(a, b)$ when there may be confusion about to which category hom(a, b) refers) to denote the **hom-class** of all morphisms from a to b. (Some authors write Mor(a, b) or simply $C(a, b)$ instead.)
- for every three objects a, b and c, a binary operation hom(a, b) × hom(b, c) → hom(a, c) called *composition of morphisms*; the composition of $f : a \to b$ and $g : b \to c$ is written as $g \circ f$ or gf. (Some authors write fg or $f;g$.)

such that the following axioms hold:

- (associativity) if $f : a \to b$, $g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
- (identity) for every object x, there exists a morphism $1_x : x \to x$ called the *identity morphism for x*, such that for every morphism $f : a \to b$, we have $1_b \circ f = f = f \circ 1_a$.

From these axioms, one can prove that there is exactly one identity morphism for every object. Some authors use a slight variation of the definition in which each object is identified with the corresponding identity morphism.

Small and large categories

A category C is called **small** if both ob(C) and hom(C) are actually sets and not proper classes, and **large** otherwise. A **locally small category** is a category such that for all objects a and b, the hom-class hom(a, b) is a set, called a **homset**. Many important categories in mathematics (such as the category of sets), although not small, are at least locally small.

Examples

The class of all sets together with all functions between sets, where composition is the usual function composition, forms a large category, **Set**. It is the most basic and the most commonly used category in mathematics. The category **Rel** consists of all sets, with binary relations as morphisms. Abstracting from relations instead of functions yields allegories instead of categories.

Any class can be viewed as a category whose only morphisms are the identity morphisms. Such categories are called discrete. For any given set I, the *discrete category on I* is the small category that has the elements of I as objects and only the identity morphisms as morphisms. Discrete categories are the simplest kind of category.

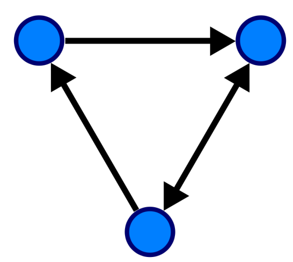
Any preordered set (P, ≤) forms a small category, where the objects are the members of P, the morphisms are arrows pointing from x to y when $x \leq y$. Between any two objects there can be at most one morphism. The existence of identity morphisms and the composability of the morphisms are guaranteed by the reflexivity and the transitivity of the preorder. By the same argument, any partially ordered set and any equivalence relation can be seen as a small category. Any ordinal number can be seen as a category when viewed as a ordered set.

Any monoid (any algebraic structure with a single associative binary operation and an identity element) forms a small category with a single object x. (Here, x is any fixed set.) The morphisms from x to x are precisely the elements of the monoid, the identity morphism of x is the identity of the monoid, and the categorical composition of morphisms is given by the monoid operation. Several definitions and theorems about monoids may be generalized for categories.

Any group can be seen as a category with a single object in which every morphism is invertible (for every morphism f there is a morphism g that is both left and right inverse to f under composition) by viewing the group as acting on itself by left multiplication. A morphism which is invertible in this sense is called an isomorphism.

A groupoid is a category in which every morphism is an isomorphism. Groupoids are generalizations of groups, group actions and equivalence relations.

Any directed graph generates a small category: the objects are the vertices of the graph, and the morphisms are the paths in the graph (augmented with loops as needed) where composition of morphisms is concatenation of paths. Such a category is called the *free category* generated by the graph.


A directed graph.

The class of all preordered sets with monotonic functions as morphisms forms a category, **Ord**. It is a concrete category, i.e. a category obtained by adding some type of structure onto **Set**, and requiring that morphisms are functions that respect this added structure.

The class of all groups with group homomorphisms as morphisms and function composition as the composition operation forms a large category, **Grp**. Like **Ord**, **Grp** is a concrete category. The category **Ab**, consisting of all abelian groups and their group homomorphisms, is a full subcategory of **Grp**, and the prototype of an abelian category. Other examples of concrete categories are given by the following table.

Category	Objects	Morphisms
Mag	magmas	magma homomorphisms
Ring	rings	ring homomorphisms
R-Mod	R-Modules, where R is a Ring	module homomorphisms
$\mathbf{Vect}_K$	vector spaces over the field K	K-linear maps
Top	topological spaces	continuous functions
Met	metric spaces	short maps
Uni	uniform spaces	uniformly continuous functions
$\mathbf{Man}^p$	smooth manifolds	p-times continuously differentiable maps

Fiber bundles with bundle maps between them form a concrete category.

The category **Cat** consists of all small categories, with functors between them as morphisms.

Construction of new categories

Dual category

Any category C can itself be considered as a new category in a different way: the objects are the same as those in the original category but the arrows are those of the original category reversed. This is called the *dual* or *opposite category* and is denoted C^{op}.

Product categories

If C and D are categories, one can form the *product category* $C \times D$: the objects are pairs consisting of one object from C and one from D, and the morphisms are also pairs, consisting of one morphism in C and one in D. Such pairs can be composed componentwise.

Types of morphisms

A morphism $f : a \to b$ is called

- a *monomorphism* (or *monic*) if $fg_1 = fg_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : x \to a$.
- an *epimorphism* (or *epic*) if $g_1 f = g_2 f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : b \to x$.
- a **bimorphism** if it is both a monomorphism and an epimorphism.
- a *retraction* if it has a right inverse, i.e. if there exists a morphism $g : b \to a$ with $fg = 1_b$.
- a *section* if it has a left inverse, i.e. if there exists a morphism $g : b \to a$ with $gf = 1_a$.
- an *isomorphism* if it has an inverse, i.e. if there exists a morphism $g : b \to a$ with $fg = 1_b$ and $gf = 1_a$.
- an *endomorphism* if $a = b$. The class of endomorphisms of a is denoted end(a).
- an *automorphism* if f is both an endomorphism and an isomorphism. The class of automorphisms of a is denoted aut(a).

Every retraction is an epimorphism. Every section is a monomorphism. The following three statements are equivalent:

- f is a monomorphism and a retraction;
- f is an epimorphism and a section;
- f is an isomorphism.

Relations among morphisms (such as $fg = h$) can most conveniently be represented with commutative diagrams, where the objects are represented as points and the morphisms as arrows.

Types of categories

- In many categories, e.g. **Ab** or $\mathbf{Vect}_K$, the hom-sets hom(a, b) are not just sets but actually abelian groups, and the composition of morphisms is compatible with these group structures; i.e. is bilinear. Such a category is called preadditive. If, furthermore, the category has all finite products and coproducts, it is called an additive category. If all morphisms have a kernel and a cokernel, and all epimorphisms are cokernels and all monomorphisms are kernels, then we speak of an abelian category. A typical example of an abelian category is the category of abelian groups.
- A category is called complete if all limits exist in it. The categories of sets, abelian groups and topological spaces are complete.
- A category is called cartesian closed if it has finite direct products and a morphism defined on a finite product can always be represented by a morphism defined on just one of the factors. Examples include **Set** and **CPO**, the category of complete partial orders with Scott-continuous functions.
- A topos is a certain type of cartesian closed category in which all of mathematics can be formulated (just like classically all of mathematics is formulated in the category of sets). A topos can also be used to represent a logical theory.

See also

- Enriched category
- Higher category theory
- Table of mathematical symbols

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External links

- Chris Hillman, Categorical primer [14], formal introduction to Category Theory.
- Homepage of the Categories mailing list [17], with extensive list of resources
- *Category Theory* section of Alexandre Stefanov's list of free online mathematics resources [3]

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Morphism

In mathematics, a **morphism** is an abstraction derived from **structure-preserving mappings** between two mathematical structures.

The study of morphisms and of the structures (called objects) over which they are defined, is central to category theory. Much of the terminology of morphisms, as well as the intuition underlying them, comes from concrete categories, where the objects are simply sets with some additional structure, and morphisms are functions preserving this structure. Nevertheless, morphisms are not necessarily functions, and objects over which morphisms are defined are not necessarily sets. Instead, a morphism is often thought of as an arrow linking an object called the domain to another object called the codomain. Hence morphisms do not so much map sets into sets, as embody a relationship between some posited domain and codomain.

The notion of morphism recurs in much of contemporary mathematics. In set theory, morphisms are functions; in linear algebra, linear transformations; in group theory, group homomorphisms; in topology, continuous functions; in universal algebra, homomorphisms.

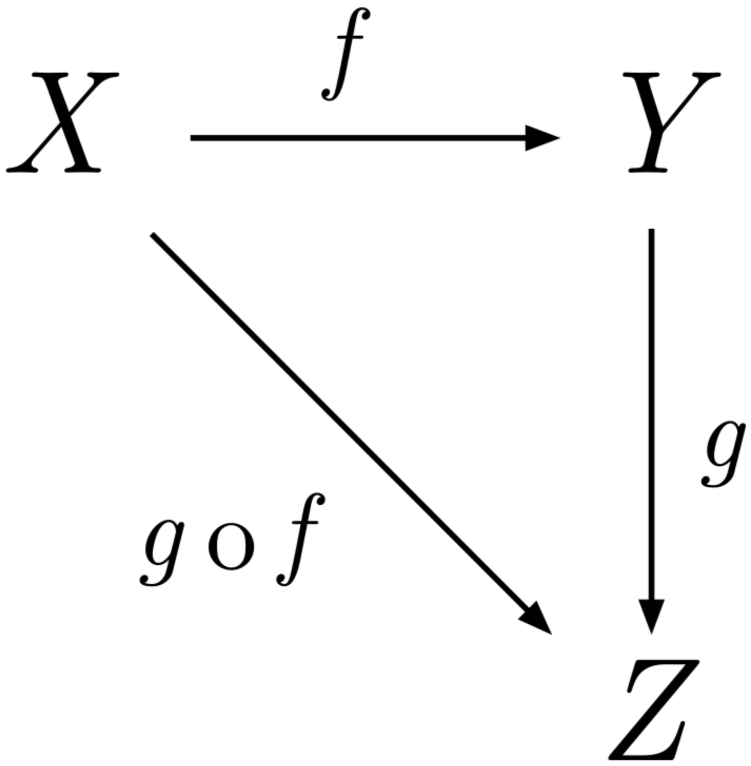
Definition

A category C consists of two classes, one of *objects* and the other of *morphisms*.

There are two operations which are defined on every morphism, the **domain** (or **source**) and the **codomain** (or **target**).

If a morphism f has domain X and codomain Y, we write $f : X \to Y$. Thus a morphism is represented by an *arrow* from its domain to its codomain. The collection of all morphisms from X to Y is denoted $\hom_C(X,Y)$ or simply $\hom(X, Y)$ and called the **hom-set** between X and Y. Some authors write $\mathrm{Mor}_C(X,Y)$ or $\mathrm{Mor}(X, Y)$. Note that the term hom-set is a bit of a misnomer as the collection of morphisms is not required to be a set.

For every three objects X, Y, and Z, there exists a binary operation $\hom(X, Y) \times \hom(Y, Z) \to \hom(X, Z)$ called composition. The composite of $f : X \to Y$ and $g : Y \to Z$ is written $g \circ f$ or gf. The composition of morphisms is often represented by a commutative diagram. For example,



Morphisms satisfy two axioms:

- *Identity:* for every object X, there exists a morphism $\mathrm{id}_X : X \to X$ called the **identity morphism** on X, such that for every morphism $f : A \to B$ we have $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.
- *Associativity:* $h \circ (g \circ f) = (h \circ g) \circ f$ whenever the operations are defined.

When C is a concrete category, the identity morphism is just the identity function, and composition is just the ordinary composition of functions. *Associativity* then follows, because the composition of functions is associative.

Note that the domain and codomain are in fact part of the information determining a morphism. For example, in the category of sets, where morphisms are functions, two functions may be identical as sets of ordered pairs (may have the same range), while having different codomains. The two functions are distinct from the viewpoint of category theory. Thus many authors require that the hom-classes hom(X, Y) be disjoint. In practice, this is not a problem because if this disjointness does not hold, it can be assured by appending the domain and codomain to the morphisms, (say, as the second and third components of an ordered triple).

Alternate definition using a "null morphism"

Since there is exactly one identity morphism id_X for each object X, the class of objects can be dropped from the definition of a category, and replaced with the subclass of hom_C consisting of the identity morphisms. In this formulation, a category C consists of a non-empty class hom_C with one additional structure: the composition function, a binary operation $\circ$: $\mathrm{hom}_C \times \mathrm{hom}_C \to \mathrm{hom}_C$. Composition is defined for *all* pairs of morphisms (elements of hom_C), with the help of a *null morphism* (or just *null*) ø in hom_C, which obeys $fø = øf = ø$ for every morphism f. The class C_0 of *identity morphisms* (or just *identities*) consists of those elements $X≠ø$ of hom_C such that, for every g in hom_C, $g \circ X \in \{g, \emptyset\}$. Up to isomorphism, the only category with no identities is the *null category* **0** = {ø} (equipped with the obvious composition function).

To form a category, the composition operation must be associative and must also split over the identity morphisms, meaning that:

- For every $f \neq \emptyset$ in $\hom_C$, fX must be non-null (and equal to f) for exactly one X in C_0 (the domain of f).
- For every $g \neq \emptyset$ in $\hom_C$, Yg must be non-null (and equal to g) for exactly one Y in C_0 (the codomain of g).
- Consequently, domain(f) = codomain(g) is a necessary condition for fg to be non-null. This must also be a sufficient condition.

Thus the class $\hom_C$ of morphisms is the union of the non-overlapping classes $\{\hom_C(X,Y), \mathbf{0}\}$. The domain $\hom_C \times \hom_C$ of the composition operation may be divided into the null sector $\circ^{-1}(\emptyset)$and the collection of non-null sectors $\hom_C(X,Y) \times \hom_C(Y,Z)$.
The two definitions of a category are equivalent, but the formulation with the "null morphism" has several advantages:

- We can identify the category with its class of morphisms (including the null morphism), and write C for both. Thus a category is simply a non-empty class C, equipped with an associative binary operation with null ø, which splits over a subclass C_0 of identities.
- The composition operation is a total function on $C \times C$; instead of splitting C into hom-classes and enumerating cases in which composition is and isn't defined, one can usually make simpler statements about the preimage of ø.
- Statements about the objects of the category reduce to statements about the subclass C_0 of identity morphisms in $\hom_C$, and can often be subsumed into facts about identity morphisms. For instance, a (total) functor F from category C to category D is just a function that preserves the composition operation (including its null and its subclass of identities). The statement that F preserves the identity subclass implies that $F(C_0) \subseteq D_0$; the mapping of objects given in the usual definition of a functor is recovered as the restriction of F to C_0.
- We obtain a null category $\mathbf{0} = \{ø\}$, distinct from the trivial category $\mathbf{1} = \{1, ø\}$. (The trivial category is equipped with the only composition operation under which 1 is not equivalent to ø.) This null category serves as a zero object in the category of small categories **Cat**, if we take the morphisms of **Cat** to include all *partial functors*: maps $F: C \to D$, which preserve the composition operation (and its null) and map C_0 into $D_0 \cup \{\emptyset\}$. (Note that "partial functor" is usually used in a different sense, analogous to the use of partial function to describe a function of several variables in which some have already been fixed.)

This version of the category of small categories is not the same as the usual definition of **Cat**, in which the class of morphisms is limited to the total functors, and thus the empty category ∅ = {} is an initial object but the terminal object is **1** = {1}. Statements about functors can be clearly divided into those that apply also to partial functors and those that apply only to total functors (those with kernel **0**). Similarly, one can define a version of the category of sets in which the morphisms are the partial functions and the null set is a zero object; the total functions are those partial functions whose kernel is the null set. These examples illustrate that the essential property of a category is not its class of objects, nor even its class of morphisms, but its composition operation. This operation is usually implicit in the name of the class of morphisms; thus it would perhaps be better to name a category after its morphisms (e. g., the "category of total functions" vs. the "category of partial functions") rather than after its objects (the "category of sets").

Some specific morphisms

- **Monomorphism**: $f : X \to Y$ is called a monomorphism if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : Z \to X$. It is also called a *mono* or a *monic*.
  - The morphism f has a **left inverse** if there is a morphism $g: Y \to X$ such that $g \circ f = \mathrm{id}_X$. The left inverse g is also called a **retraction** of f. Morphisms with left inverses are always monomorphisms, but the converse is not always true in every category; a monomorphism may fail to have a left-inverse.
  - A **split monomorphism** $h : X \to Y$ is a monomorphism having a left inverse $g : Y \to X$, so that $g \circ h = \mathrm{id}_X$. Thus $h \circ g : Y \to Y$ is idempotent, so that $(h \circ g)^2 = h \circ g$.
  - In concrete categories, a function that has left inverse is injective. Thus in concrete categories, monomorphisms are often, but not always, injective. The condition of being an injection is stronger than that

of being a monomorphism, but weaker than that of being a split monomorphism.

- **Epimorphism**: Dually, $f : X \to Y$ is called an epimorphism if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all morphisms g_1, $g_2 : Y \to Z$. It is also called an *epi* or an *epic*.
  - The morphism f has a **right-inverse** if there is a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$. The right inverse g is also called a **section** of f. Morphisms having a right inverse are always epimorphisms, but the converse is not always true in every category, as an epimorphism may fail to have a right inverse.
  - A **split epimorphism** is an epimorphism having a right inverse. Note that if a monomorphism f splits with left-inverse g, then g is a split epimorphism with right-inverse f.
  - In concrete categories, a function that has a right inverse is surjective. Thus in concrete categories, epimorphisms are often, but not always, surjective. The condition of being a surjection is stronger than that of being an epimorphism, but weaker than that of being a split epimorphism. In the category of sets, every surjection has a section, a result equivalent to the axiom of choice.
- A **bimorphism** is a morphism that is both an epimorphism and a monomorphism.
- **Isomorphism**: $f : X \to Y$ is called an isomorphism if there exists a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. If a morphism has both left-inverse and right-inverse, then the two inverses are equal, so f is an isomorphism, and g is called simply the **inverse** of f. Inverse morphisms, if they exist, are unique. The inverse g is also an isomorphism with inverse f. Two objects with an isomorphism between them are said to be isomorphic or equivalent. Note that while every isomorphism is a bimorphism, a bimorphism is not necessarily an isomorphism. For example, in the category of commutative rings the inclusion $\mathbf{Z} \to \mathbf{Q}$ is a bimorphism, which is not an isomorphism. However, any morphism that is both an epimorphism and a *split* monomorphism, or both a monomorphism and a *split* epimorphism, must be an isomorphism. A category, such as **Set**, in which every bimorphism is an isomorphism is known as a **balanced category**.
- **Endomorphism**: $f : X \to X$ is an endomorphism of X. A **split endomorphism** is an idempotent endomorphism f if f admits a decomposition $f = h \circ g$ with $g \circ h = \mathrm{id}$. In particular, the Karoubi envelope of a category splits every idempotent morphism.
- An **automorphism** is a morphism that is both an endomorphism and an isomorphism.

Examples

- In the concrete categories studied in universal algebra (groups, rings, modules, etc.), morphisms are called homomorphisms. Likewise, the notions of automorphism, endomorphism, epimorphism, homeomorphism, isomorphism, and monomorphism all find use in universal algebra.
- In the category of topological spaces, morphisms are continuous functions and isomorphisms are called homeomorphisms.
- In the category of smooth manifolds, morphisms are smooth functions and isomorphisms are called diffeomorphisms.
- In the category of small categories, functors can be thought of as morphisms.
- In a functor category, the morphisms are natural transformations.

For more examples, see the entry category theory.

See also

- anamorphism
- automorphism
- catamorphism
- category theory
- concrete category
- diffeomorphism
- endomorphism
- epimorphism
- holomorphic function
- homeomorphism
- homomorphism
- hylomorphism
- isomorphism
- monomorphism
- normal morphism
- paramorphism
- zero morphism

External links

- Category [1] on PlanetMath
- TypesOfMorphisms [2] on PlanetMath

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[2] http://planetmath.org/?op=getobj&from=objects&id=8114

Isomorphism

In abstract algebra, an **isomorphism** (Greek: ἴσος *isos* "equal", and μορφή *morphe* "shape") is a bijective map f such that both f and its inverse f^{-1} are homomorphisms, i.e., *structure-preserving* mappings. In the more general setting of category theory, an **isomorphism** is a morphism f: $X \to Y$ in a category for which there exists an "inverse" f^{-1}: $Y \to X$, with the property that both $f^{-1}f = \mathrm{id}_X$ and $f f^{-1} = \mathrm{id}_Y$.

Informally, an isomorphism is a kind of mapping between objects, which shows a relationship between two properties or operations. If there exists an isomorphism between two structures, we call the two structures **isomorphic**. In a certain sense, isomorphic structures are **structurally identical**, if you choose to ignore finer-grained differences that may arise from how they are defined.

Purpose

Isomorphisms are studied in mathematics in order to extend insights from one phenomenon to others: if two objects are isomorphic, then any property which is preserved by an isomorphism and which is true of one of the objects, is also true of the other. If an isomorphism can be found from a relatively unknown part of mathematics into some well studied division of mathematics, where many theorems are already proved, and many methods are already available to find answers, then the function can be used to map whole problems out of unfamiliar territory over to "solid ground" where the problem is easier to understand and work with.

Practical example

The following are examples of isomorphisms from ordinary algebra.

- Consider the logarithm function: For any fixed base b, the logarithm function $\log_b$ maps from the positive real numbers $\mathbb{R}^+$ onto the real numbers $\mathbb{R}$; formally:

  $$\log_b : \mathbb{R}^+ \to \mathbb{R}.$$

  This mapping is one-to-one and onto, that is, it is a bijection from the domain to the codomain of the logarithm function. In addition to being an isomorphism of sets, the logarithm function also preserves certain operations. Specifically, consider the group $(\mathbb{R}^+, \times)$ of positive real numbers under ordinary multiplication. The logarithm function obeys the following identity:

  $$\log_b(xy) = \log_b(x) + \log_b(y).$$

  But the real numbers under addition also form a group. So the logarithm function is in fact a group isomorphism from the group $(\mathbb{R}^+, \times)$ to the group $(\mathbb{R}, +)$.

Logarithms can therefore be used to simplify multiplication of real numbers. By working with logarithms, multiplication of positive real numbers is replaced by addition of logs. This way it is possible to multiply real numbers using a ruler and a table of logarithms, or using a slide rule with a logarithmic scale.

- Consider the group $\mathbf{Z}_6$, the numbers from 0 to 5 with addition modulo 6. Also consider the group $\mathbf{Z}_2 \times \mathbf{Z}_3$, the ordered pairs where the x coordinates can be 0 or 1, and the y coordinates can be 0, 1, or 2, where addition in the x-coordinate is modulo 2 and addition in the y-coordinate is modulo 3. These structures are isomorphic under addition, if you identify them using the following scheme:

  (0,0) → 0

  (1,1) → 1

  (0,2) → 2

  (1,0) → 3

  (0,1) → 4

$(1,2) \rightarrow 5$

or in general $(a,b) \rightarrow (3a + 4b) \bmod 6$. For example note that (1,1) + (1,0) = (0,1) which translates in the other system as 1 + 3 = 4. Even though these two groups "look" different in that the sets contain different elements, they are indeed **isomorphic**: their structures are exactly the same. More generally, the direct product of two cyclic groups $\mathbf{Z}_n$ and $\mathbf{Z}_m$ is cyclic if and only if n and m are coprime.

Abstract examples

A relation-preserving isomorphism

If one object consists of a set X with a binary relation R and the other object consists of a set Y with a binary relation S then an isomorphism from X to Y is a bijective function $f : X \rightarrow Y$ such that

$f(u)$ S $f(v)$ if and only if u R v.

S is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, transitive, total, trichotomous, a partial order, total order, strict weak order, total preorder (weak order), an equivalence relation, or a relation with any other special properties, if and only if R is.

For example, R is an ordering ≤ and S an ordering $\sqsubseteq$, then an isomorphism from X to Y is a bijective function $f : X \rightarrow Y$ such that

$f(u) \sqsubseteq f(v)$ if and only if $u \le v$.

Such an isomorphism is called an *order isomorphism* or (less commonly) an *isotone isomorphism*.

If $X = Y$ we have a relation-preserving automorphism.

An operation-preserving isomorphism

Suppose that on these sets X and Y, there are two binary operations $\star$ and $\Diamond$ which happen to constitute the groups $(X, \star)$ and $(Y, \Diamond)$. Note that the operators operate on elements from the domain and range, respectively, of the "one-to-one" and "onto" function f. There is an isomorphism from X to Y if the bijective function $f : X \rightarrow Y$ happens to produce results, that sets up a correspondence between the operator $\star$ and the operator $\Diamond$.

$$f(u) \Diamond f(v) = f(u \star v)$$

for all u, v in X.

Applications

In abstract algebra, two basic isomorphisms are defined:

- Group isomorphism, an isomorphism between groups
- Ring isomorphism, an isomorphism between rings. (Note that isomorphisms between fields are actually ring isomorphisms)

Just as the automorphisms of an algebraic structure form a group, the isomorphisms between two algebras sharing a common structure form a heap. Letting a particular isomorphism identify the two structures turns this heap into a group.

In mathematical analysis, the Laplace transform is an isomorphism mapping hard differential equations into easier algebraic equations.

In category theory, Iet the category C consist of two classes, one of *objects* and the other of morphisms. Then a general definition of isomorphism that covers the previous and many other cases is: an isomorphism is a morphism $f : a \rightarrow b$ that has an inverse, i.e. there exists a morphism $g : b \rightarrow a$ with $fg = 1_b$ and $gf = 1_a$. For example, a bijective linear map is an isomorphism between vector spaces, and a bijective continuous function whose inverse is also continuous is an isomorphism between topological spaces, called a homeomorphism.

In graph theory, an isomorphism between two graphs *G* and *H* is a bijective map *f* from the vertices of *G* to the vertices of *H* that preserves the "edge structure" in the sense that there is an edge from vertex *u* to vertex *v* in *G* if and only if there is an edge from *f*(*u*) to *f*(*v*) in *H*. See graph isomorphism.

In early theories of logical atomism, the formal relationship between facts and true propositions was theorized by Bertrand Russell and Ludwig Wittgenstein to be isomorphic. An example of this line of thinking can be found in Russell's Introduction to Mathematical Philosophy.

In cybernetics, the Good Regulator or Conant-Ashby theorem is stated "Every Good Regulator of a system must be a model of that system". Whether regulated or self-regulating an isomorphism is required between regulator part and the processing part of the system.

Relation with equality

In certain areas of mathematics, notably category theory, it is very valuable to distinguish between *equality* on the one hand and *isomorphism* on the other.[1] Equality is when two objects are "literally the same", while isomorphism is when two objects "can be made to correspond via an isomorphism". For example, the sets $A = \{x \in \mathbf{Z} \mid x^2 < 2\}$ and $B = \{-1, 0, 1\}$ are *equal* – they are two different presentations (one in set builder notation, one by an enumeration) of the same subset of the integers. By contrast, the sets $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ and $\{1, 2, 3\}$ are not *equal* – the first has elements that are letters, while the second has elements that are numbers. These are isomorphic as sets, since finite sets are determined up to isomorphism by their cardinality (number of elements) and these both have three elements, but there are many choices of isomorphism – one isomorphism is $\mathrm{A} \mapsto 1, \mathrm{B} \mapsto 2, \mathrm{C} \mapsto 3,$ while another is $\mathrm{A} \mapsto 3, \mathrm{B} \mapsto 2, \mathrm{C} \mapsto 1,$ and no one isomorphism is better than any other.[2] [3] Thus one cannot *identify* these two sets: one can choose an isomorphism between them, but any statement that identifies these two sets *depends* on the choice of isomorphism.

A motivating example of this distinction is the distinction between a finite-dimensional vector space *V* and its dual space V^*. These spaces have the same dimension, and thus are isomorphic as abstract vector spaces (since algebraically, vector spaces are classified by dimension, just as sets are classified by cardinality), but there is no "natural" choice of isomorphism $V \xrightarrow{\sim} V^*$. If one chooses a basis for *V,* then this yields an isomorphism; this corresponds to transforming a column vector (element of *V*) to a row vector (element of V^*) by transpose, but a different choice of basis gives a different isomorphism: the isomorphism "depends on the choice of basis". More subtly, there *is* a map from a vector space to its double dual $V \to V^{**}$ that does not depend on the choice of basis. This leads to a third notion, that of a natural isomorphism: while V and V^{**} are different sets (the first consists of vectors *v* ($V = \{v\}$), while the latter consists of functions from the dual space V^* to the field *K* of scalars ($V^{**} = \{\phi\colon V^* \to K\}$)), there is a "natural" choice of isomorphism between them. This intuitive notion of "an isomorphism that does not depend on an arbitrary choice" is formalized in the notion of a natural transformation: briefly, that one may *consistently* identify, or more generally map from, a vector space to its double dual, across *all* vector spaces: one may map $V \xrightarrow{\sim} V^{**}$ and $W \xrightarrow{\sim} W^{**}$ and so forth for all vector space in a consistent way. Formalizing this intuition is a motivation for the development of category theory.

If one wishes to draw a distinction between an arbitrary isomorphism (one that depends on a choice) and a natural isomorphism (one that can be done consistently), one may write $\approx$ for an unnatural isomorphism and $\cong$ for a natural isomorphism, as in $V \approx V^*$ and $V \cong V^{**}$; this convention is not universally followed, and authors who wish to distinguish between unnatural isomorphisms and natural isomorphisms will generally explicitly state the distinction.

Generally, saying that two objects are *equal* is reserved for when there is a notion of a larger (ambient) space which these objects live within. Most often, one speaks of equality of two subsets of a given set (as in the integer set example above), but not of two objects abstractly presented. For example, the 2-dimensional unit sphere in 3-dimensional space $S^2 := \{(x, y, z) \in \mathbf{R}^3 | x^2 + y^2 + z^2 = 1\}$ and the Riemann sphere $\widehat{\mathbf{C}}$ – which can be presented as the one-point compactification of the complex plane $\mathbf{C} \cup \{\infty\}$ *or* as the complex projective line (a quotient space) $\mathbf{P}^1_{\mathbf{C}} := (\mathbf{C}^2 \setminus \{(0,0)\})/(\mathbf{C}^*)$ – are three different descriptions for a mathematical object, all of which are isomorphic, but which are not *equal* because they are not all subsets of a single space: the first is a subset of $\mathbf{R}^3$, the second is $\mathbf{C} \cong \mathbf{R}^2$[4] plus an additional point, and the third is a subquotient of $\mathbf{C}^2$.

See also

- Epimorphism
- Heap (mathematics)
- Isometry
- Isomorphism class
- Monomorphism
- Morphism

References

[1] (Mazur 2007)

[2] The careful reader may note that A, B, C have a conventional order, namely alphabetical order, and similarly 1, 2, 3 have the order from the integers, and thus in some sense there is a "natural" isomorphism, namely $\mathrm{A} \mapsto 1, \mathrm{B} \mapsto 2, \mathrm{C} \mapsto 3.$ This intuition can be formalized by saying that any two finite *ordered* sets of the same cardinality have a natural isomorphism, by sending the least element of the first to the least element of the second, the least element of what remains in the first to the least element of what remains in the second, and so forth, but conversely two finite *unordered* sets are not naturally isomorphic because there is more than one choice of map – except if the cardinality is 0 or 1, where there is a unique choice.

[3] In fact, there are precisely $3! = 6$ different isomorphism between two sets with three elements. This is equal to the number of automorphisms of a given three element set (the symmetric group on three letters), and more generally one has that the set of isomorphisms between two objects, denoted $\mathrm{Iso}(A, B)$, is a torsor for the automorphism group of *A*, $\mathrm{Aut}(A)$ and also a torsor for the automorphism group of *B*. In fact, automorphisms of an object are a key reason to be concerned with the distinction between isomorphism and equality, as demonstrated in the effect of change of basis on the identification of a vector space with its dual or with its double dual, as elaborated in the sequel.

[4] Being precise, the identification of the complex numbers with the real plane, $\mathbf{C} \cong \mathbf{R} \cdot 1 \oplus \mathbf{R} \cdot i = \mathbf{R}^2$ depends on a choice of i; one can just as easily choose $(-i)$, which yields a different identification – formally, complex conjugation is an automorphism – but in practice one often assumes that one has made such an identification.

- Mazur, Barry (12 June 2007), *When is one thing equal to some other thing?* (http://www.math.harvard.edu/~mazur/preprints/when_is_one.pdf)

External links

- Isomorphism (http://planetmath.org/?op=getobj&from=objects&id=1936) on PlanetMath
- Weisstein, Eric W., " Isomorphism (http://mathworld.wolfram.com/Isomorphism.html)" from MathWorld.

Functor

In category theory, a branch of mathematics, a **functor** is a special type of mapping between categories. Functors can be thought of as homomorphisms between categories, or morphisms in the category of small categories.

Functors were first considered in algebraic topology, where algebraic objects (like the fundamental group) are associated to topological spaces, and algebraic homomorphisms are associated to continuous maps. Nowadays, functors are used throughout modern mathematics to relate various categories. The word "functor" was borrowed by mathematicians from the philosopher Rudolf Carnap [Mac Lane, p. 30]. Carnap used the term "functor" to stand in relation to functions analogously as predicates stand in relation to properties. [See Carnap, The Logical Syntax of Language, p.13-14, 1937, Routledge & Kegan Paul.] For Carnap then, unlike modern category theory's use of the term, a functor is a linguistic item. For category theorists, a functor is a particular kind of function.

Definition

Let *C* and *D* be categories. A **functor** *F* from *C* to *D* is a mapping that

- associates to each object $X \in C$ an object $F(X) \in D$,
- associates to each morphism $f : X \to Y \in C$ a morphism $F(f) : F(X) \to F(Y) \in D$

such that the following two conditions hold:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object $X \in C$
- $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$.

That is, functors must preserve identity morphisms and composition of morphisms.

Covariance and contravariance

There are many constructions in mathematics which would be functors but for the fact that they "turn morphisms around" and "reverse composition". We then define a **contravariant functor** *F* from *C* to *D* as a mapping that

- associates to each object $X \in C$ an object $F(X) \in D$,
- associates to each morphism $f : X \to Y \in C$ a morphism $F(f) : F(Y) \to F(X) \in D$ such that
  - $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object $X \in C$,
  - $F(g \circ f) = F(f) \circ F(g)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$.

Note that contravariant functors reverse the direction of composition.

Ordinary functors are also called **covariant functors** in order to distinguish them from contravariant ones. Note that one can also define a contravariant functor as a *covariant* functor on the dual category C^{op} . Some authors prefer to write all expressions covariantly. That is, instead of saying $F : C \to D$ is a contravariant functor, they simply write $F : C^{\mathrm{op}} \to D$ (or sometimes $F : C \to D^{\mathrm{op}}$) and call it a functor.

Contravariant functors are also occasionally called *cofunctors*.

Examples

Constant functor: The functor $C \to D$ which maps every object of C to a fixed object X in D and every morphism in C to the identity morphism on X. Such a functor is called a *constant* or *selection* functor.

Endofunctor: A functor that maps a category to itself.

Diagonal functor: The diagonal functor is defined as the functor from D to the functor category D^C which sends each object in D to the constant functor at that object.

Limit functor: For a fixed index category J, if every functor $J \to C$ has a limit (for instance if C is complete), then the limit functor $C^J \to C$ assigns to each functor its limit. The existence of this functor can be proved by realizing that it is the right-adjoint to the diagonal functor and invoking the Freyd adjoint functor theorem. This requires a suitable version of the axiom of choice. Similar remarks apply to the colimit functor (which is covariant).

Power sets: The power set functor P : **Set** → **Set** maps each set to its power set and each function $f : X \to Y$ to the map which sends $U \subseteq X$ to its image $f(U) \subseteq Y$. One can also consider the contravariant power set functor which sends $f : X \to Y$ to the map which sends $V \subseteq Y$ to its inverse image $f^{-1}(V) \subseteq X$.

Dual vector space: The map which assigns to every vector space its dual space and to every linear map its dual or transpose is a contravariant functor from the category of all vector spaces over a fixed field to itself.

Fundamental group: Consider the category of pointed topological spaces, i.e. topological spaces with distinguished points. The objects are pairs (X, x_0), where X is a topological space and x_0 is a point in X. A morphism from (X, x_0) to (Y, y_0) is given by a continuous map $f : X \to Y$ with $f(x_0) = y_0$.

To every topological space X with distinguished point x_0, one can define the fundamental group based at x_0, denoted $\pi_1(X, x_0)$. This is the group of homotopy classes of loops based at x_0. If $f : X \to Y$ morphism of pointed spaces, then every loop in X with base point x_0 can be composed with f to yield a loop in Y with base point y_0. This operation is compatible with the homotopy equivalence relation and the composition of loops, and we get a group homomorphism from $\pi(X, x_0)$ to $\pi(Y, y_0)$. We thus obtain a functor from the category of pointed topological spaces to the category of groups.

In the category of topological spaces (without distinguished point), one considers homotopy classes of generic curves, but they cannot be composed unless they share an endpoint. Thus one has the **fundamental groupoid** instead of the fundamental group, and this construction is functorial.

Algebra of continuous functions: a contravariant functor from the category of topological spaces (with continuous maps as morphisms) to the category of real associative algebras is given by assigning to every topological space X the algebra C(X) of all real-valued continuous functions on that space. Every continuous map $f : X \to Y$ induces an algebra homomorphism $C(f) : C(Y) \to C(X)$ by the rule $C(f)(\varphi) = \varphi \circ f$ for every φ in C(Y).

Tangent and cotangent bundles: The map which sends every differentiable manifold to its tangent bundle and every smooth map to its derivative is a covariant functor from the category of differentiable manifolds to the category of vector bundles. Likewise, the map which sends every differentiable manifold to its cotangent bundle and every smooth map to its pullback is a contravariant functor.

Doing these constructions pointwise gives covariant and contravariant functors from the category of pointed differentiable manifolds to the category of real vector spaces.

Group actions/representations: Every group G can be considered as a category with a single object whose morphisms are the elements of G. A functor from G to **Set** is then nothing but a group action of G on a particular set, i.e. a G-set. Likewise, a functor from G to the category of vector spaces, $\mathbf{Vect}_K$, is a linear representation of G. In general, a functor $G \to C$ can be considered as an "action" of G on an object in the category C. If C is a group, then this action is a group homomorphism.

Lie algebras: Assigning to every real (complex) Lie group its real (complex) Lie algebra defines a functor.

Tensor products: If C denotes the category of vector spaces over a fixed field, with linear maps as morphisms, then the tensor product $V \otimes W$ defines a functor $C \times C \to C$ which is covariant in both arguments.

Forgetful functors: The functor U : **Grp** → **Set** which maps a group to its underlying set and a group homomorphism to its underlying function of sets is a functor. Functors like these, which "forget" some structure, are termed *forgetful functors*. Another example is the functor **Rng** → **Ab** which maps a ring to its underlying additive abelian group. Morphisms in **Rng** (ring homomorphisms) become morphisms in **Ab** (abelian group homomorphisms).

Free functors: Going in the opposite direction of forgetful functors are free functors. The free functor F : **Set** → **Grp** sends every set X to the free group generated by X. Functions get mapped to group homomorphisms between free groups. Free constructions exist for many categories based on structured sets. See free object.

Homomorphism groups: To every pair A, B of abelian groups one can assign the abelian group Hom(A,B) consisting of all group homomorphisms from A to B. This is a functor which is contravariant in the first and covariant in the second argument, i.e. it is a functor $\mathbf{Ab}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab}$ (where **Ab** denotes the category of abelian groups with group homomorphisms). If $f : A_1 \to A_2$ and $g : B_1 \to B_2$ are morphisms in **Ab**, then the group homomorphism $\mathrm{Hom}(f,g) : \mathrm{Hom}(A_2,B_1) \to \mathrm{Hom}(A_1,B_2)$ is given by $\varphi \mapsto g \circ \varphi \circ f$. See Hom functor.

Representable functors: We can generalize the previous example to any category C. To every pair X, Y of objects in C one can assign the set Hom(X,Y) of morphisms from X to Y. This defines a functor to **Set** which is contravariant in the first argument and covariant in the second, i.e. it is a functor $C^{\mathrm{op}} \times C \to \mathbf{Set}$. If $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ are morphisms in C, then the group homomorphism $\mathrm{Hom}(f,g) : \mathrm{Hom}(X_2,Y_1) \to \mathrm{Hom}(X_1,Y_2)$ is given by $\varphi \mapsto g \circ \varphi \circ f$.

Functors like these are called representable functors. An important goal in many settings is to determine whether a given functor is representable.

Presheaves: If X is a topological space, then the open sets in X form a partially ordered set Open(X) under inclusion. Like every partially ordered set, Open(X) forms a small category by adding a single arrow $U \to V$ if and only if $U \subseteq V$. Contravariant functors on Open(X) are called *presheaves* on X. For instance, by assigning to every open set U the associative algebra of real-valued continuous functions on U, one obtains a presheaf of algebras on X.

Properties

Two important consequences of the functor axioms are:

- F transforms each commutative diagram in C into a commutative diagram in D;
- if f is an isomorphism in C, then $F(f)$ is an isomorphism in D.

On any category C one can define the **identity functor** 1_C which maps each object and morphism to itself. One can also compose functors, i.e. if F is a functor from A to B and G is a functor from B to C then one can form the composite functor GF from A to C. Composition of functors is associative where defined. This shows that functors can be considered as morphisms in categories of categories, for example in the category of small categories.

A small category with a single object is the same thing as a monoid: the morphisms of a one-object category can be thought of as elements of the monoid, and composition in the category is thought of as the monoid operation. Functors between one-object categories correspond to monoid homomorphisms. So in a sense, functors between arbitrary categories are a kind of generalization of monoid homomorphisms to categories with more than one object.

Bifunctors and multifunctors

A **bifunctor** (also known as a **binary functor**) is a functor in *two* arguments. The Hom functor is a natural example; it is contravariant in one argument, covariant in the other.

Formally, a bifunctor is a functor whose domain is a product category. For example, the Hom functor is of the type $C^{op} \times C \to \mathbf{Set}$.

A **multifunctor** is a generalization of the functor concept to n variables. So, for example, a bifunctor is a multifunctor with n=2.

Relation to other categorical concepts

Let C and D be categories. The collection of all functors $C \to D$ form the objects of a category: the functor category. Morphisms in this category are natural transformations between functors.

Functors are often defined by universal properties; examples are the tensor product, the direct sum and direct product of groups or vector spaces, construction of free groups and modules, direct and inverse limits. The concepts of limit and colimit generalize several of the above.

Universal constructions often give rise to pairs of adjoint functors.

See also

Types of functors

- Additive functor: a functor between categories whose hom-sets are abelian groups is additive if it is a group homomorphism of the hom-sets
- Adjoint functors: functors F and G are adjoint if Hom(FX,Y)≅Hom(X,GY), where the isomorphism is natural in X and Y
- Derived functor: the image of a short exact sequence under a functor that is only half-exact can be extended to a long exact sequence, the objects of which are images of a derived functor
- Enriched functor
- Essentially surjective functor: a functor every object of whose codomain is isomorphic to the image of an object in the domain
- Exact functor: a functor that takes short exact sequences to short exact sequences
- Faithful functor: a functor that is injective on the set of morphisms with given domain and codomain
- Full functor: a functor that is surjective on the set of morphisms with given domain and codomain
- Smooth functor: a functor F from K-**Vect** to K-**Vect** such that $\mathrm{Hom}(V,W) \to \mathrm{Hom}(FV,FW)$ is smooth. Examples include V^*, $\Lambda^k V$, $\Sigma^k V$ and the like.

Other

- Diagram (category theory)
- Functor category
- Kan extension

References

- S. Mac Lane. *Categories for the Working Mathematician.* Springer-Verlag: New York, 1971.

Natural transformation

In category theory, a branch of mathematics, a **natural transformation** provides a way of transforming one functor into another while respecting the internal structure (i.e. the composition of morphisms) of the categories involved. Hence, a natural transformation can be considered to be a "morphism of functors". Indeed this intuition can be formalized to define so-called functor categories. Natural transformations are, after categories and functors, one of the most basic notions of category theory and consequently appear in the majority of its applications.

Definition

If F and G are functors between the categories C and D, then a **natural transformation** η from F to G associates to every object X in C a morphism $\eta_X : F(X) \to G(X)$ in D called the **component** of η at X, such that for every morphism $f : X \to Y$ in C we have:

$$\eta_Y \circ F(f) = G(f) \circ \eta_X$$

This equation can conveniently be expressed by the commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

If both F and G are contravariant, the horizontal arrows in this diagram are reversed. If η is a natural transformation from F to G, we also write $\eta : F \to G$ or $\eta : F \Rightarrow G$. This is also expressed by saying the family of morphisms $\eta_X : F(X) \to G(X)$ is **natural** in X.

If, for every object X in C, the morphism η_X is an isomorphism in D, then η is said to be a **natural isomorphism** (or sometimes **natural equivalence** or **isomorphism of functors**). Two functors F and G are called *naturally isomorphic* or simply *isomorphic* if there exists a natural isomorphism from F to G.

An **infranatural transformation** η from F to G is simply a family of morphisms η_X: $F(X) \to G(X)$. Thus a natural transformation is an infranatural transformation for which $\eta_Y \circ F(f) = G(f) \circ \eta_X$ for every morphism $f : X \to Y$. The **naturalizer** of η, nat(η), is the largest subcategory of C containing all the objects of C on which η restricts to a natural transformation.

Examples

A worked example

Statements such as

"Every group is naturally isomorphic to its opposite group"

abound in modern mathematics. We will now give the precise meaning of this statement as well as its proof. Consider the category **Grp** of all groups with group homomorphisms as morphisms. If $(G,*)$ is a group, we define its opposite group $(G^{op},*^{op})$ as follows: G^{op} is the same set as G, and the operation $*^{op}$ is defined by $a*^{op}b = b*a$. All multiplications in G^{op} are thus "turned around". Forming the opposite group becomes a (covariant!) functor from **Grp** to **Grp** if we define $f^{op} = f$ for any group homomorphism f: $G \to H$. Note that f^{op} is indeed a group homomorphism from G^{op} to H^{op}:

$$f^{op}(a*^{op}b) = f(b*a) = f(b)*f(a) = f^{op}(a)*^{op}f^{op}(b).$$

The content of the above statement is:

"The identity functor $\mathrm{Id}_{\mathbf{Grp}}$: **Grp** → **Grp** is naturally isomorphic to the opposite functor $-^{\mathrm{op}}$: **Grp** → **Grp**."

To prove this, we need to provide isomorphisms $\eta_G : G \to G^{\mathrm{op}}$ for every group G, such that the above diagram commutes. Set $\eta_G(a) = a^{-1}$. The formulas $(ab)^{-1} = b^{-1}\ a^{-1}$ and $(a^{-1})^{-1} = a$ show that η_G is a group homomorphism which is its own inverse. To prove the naturality, we start with a group homomorphism $f : G \to H$ and show $\eta_H \circ f = f^{\mathrm{op}} \circ \eta_G$, i.e. $(f(a))^{-1} = f^{\mathrm{op}}(a^{-1})$ for all a in G. This is true since $f^{\mathrm{op}} = f$ and every group homomorphism has the property $(f(a))^{-1} = f(a^{-1})$.

Further examples

If K is a field, then for every vector space V over K we have a "natural" injective linear map $V \to V^{**}$ from the vector space into its double dual. These maps are "natural" in the following sense: the double dual operation is a functor, and the maps are the components of a natural transformation from the identity functor to the double dual functor.

Every finite dimensional vector space is also isomorphic to its dual space. But this isomorphism relies on an arbitrary choice of basis, and is not natural, though there is an infranatural transformation. More generally, any vector spaces with the same dimensionality are isomorphic, but not naturally so. (Note however that if the space has a nondegenerate bilinear form, then there *is* a natural isomorphism between the space and its dual. Here the space is viewed as an object in the category of vector spaces and transposes of maps.)

Consider the category **Ab** of abelian groups and group homomorphisms. For all abelian groups X, Y and Z we have a group isomorphism

$$\mathrm{Hom}(X \otimes Y, Z) \to \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)).$$

These isomorphisms are "natural" in the sense that they define a natural transformation between the two involved functors $\mathbf{Ab} \times \mathbf{Ab}^{\mathrm{op}} \times \mathbf{Ab}^{\mathrm{op}} \to \mathbf{Ab}$.

Natural transformations arise frequently in conjunction with adjoint functors. Indeed, adjoint functors are defined by a certain natural isomorphism. Additionally, every pair of adjoint functors comes equipped with two natural transformations (generally not isomorphisms) called the *unit* and *counit*.

Operations with natural transformations

If $\eta : F \to G$ and $\varepsilon : G \to H$ are natural transformations between functors $F,G,H:C \to D$, then we can compose them to get a natural transformation $\varepsilon\eta : F \to H$. This is done componentwise: $(\varepsilon\eta)_X = \varepsilon_X \eta_X$. This "vertical composition" of natural transformation is associative and has an identity, and allows one to consider the collection of all functors $C \to D$ itself as a category (see below under Functor categories).

Natural transformations also have a "horizontal composition". If $\eta : F \to G$ is a natural transformation between functors $F,G:C \to D$ and $\varepsilon : J \to K$ is a natural transformation between functors $J, K:D \to E$, then the composition of functors allows a composition of natural transformations $\eta\varepsilon : JF \to KG$. This operation is also associative with identity, and the identity coincides with that for vertical composition. The two operations are related by an identity which exchanges vertical composition with horizontal composition.

If $\eta : F \to G$ is a natural transformation between functors F, $G : C \to D$, and $H : D \to E$ is another functor, then we can form the natural transformation $H\eta : HF \to HG$ by defining

$$(H\eta)_X = H_{\eta_X}.$$

If on the other hand $K : B \to C$ is a functor, the natural transformation $\eta K : FK \to GK$ is defined by

$$(\eta K)_X = \eta_{K(X)}.$$

Functor categories

If C is any category and I is a small category, we can form the functor category C^I having as objects all functors from I to C and as morphisms the natural transformations between those functors. This forms a category since for any functor F there is an identity natural transformation $1_F : F \to F$ (which assigns to every object X the identity morphism on $F(X)$) and the composition of two natural transformations (the "vertical composition" above) is again a natural transformation.

The isomorphisms in C^I are precisely the natural isomorphisms. That is, a natural transformation $\eta : F \to G$ is a natural isomorphism if and only if there exists a natural transformation $\varepsilon : G \to F$ such that $\eta\varepsilon = 1_G$ and $\varepsilon\eta = 1_F$.

The functor category C^I is especially useful if I arises from a directed graph. For instance, if I is the category of the directed graph • → •, then C^I has as objects the morphisms of C, and a morphism between $\varphi : U \to V$ and $\psi : X \to Y$ in C^I is a pair of morphisms $f : U \to X$ and $g : V \to Y$ in C such that the "square commutes", i.e. $\psi f = g \varphi$.

More generally, one can build the 2-category **Cat** whose

- 0-cells (objects) are the small categories,
- 1-cells (arrows) between two objects C and D are the functors from C to D ,
- 2-cells between two 1-cells (functors) $F : C \to D$ and $G : C \to D$ are the natural transformations from F to G .

The horizontal and vertical compositions are the compositions between natural transformations described previously. A functor category C^I is then simply a hom-category in this category (smallness issues aside).

Yoneda lemma

If X is an object of a locally small category C, then the assignment $Y \mapsto \mathrm{Hom}_C(X, Y)$ defines a covariant functor $F_X : C \to$ **Set**. This functor is called *representable* (more generally, a representable functor is any functor naturally isomorphic to this functor for an appropriate choice of X). The natural transformations from a representable functor to an arbitrary functor $F : C \to$ **Set** are completely known and easy to describe; this is the content of the Yoneda lemma.

Historical notes

Saunders Mac Lane, one of the founders of category theory, is said to have remarked, "I didn't invent categories to study functors; I invented them to study natural transformations." Just as the study of groups is not complete without a study of homomorphisms, so the study of categories is not complete without the study of functors. The reason for Mac Lane's comment is that the study of functors is itself not complete without the study of natural transformations.

The context of Mac Lane's remark was the axiomatic theory of homology. Different ways of constructing homology could be shown to coincide: for example in the case of a simplicial complex the groups defined directly, and those of the singular theory, would be isomorphic. What cannot easily be expressed without the language of natural transformations is how homology groups are compatible with morphisms between objects, and how two equivalent homology theories not only have the same homology groups, but also the same morphisms between those groups.

References

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Categorical algebra

In category theory, a field of mathematics, a **categorical algebra** is an associative algebra, defined for any locally finite category and commutative ring with unity. It generalizes the notions of group algebra and incidence algebra, just as category generalizes the notions of group and partially ordered set.

Definition

Infinite categories are conventionally treated differently for group algebras and incidence algebras; the definitions agree for finite categories. We first present the definition that generalizes the group algebra.

Group algebra-style definition

Let *C* be a category and *R* be a commutative ring with unit. Then as a set and as a module, the categorical algebra *RC* (or *R*[*C*]) is the free module on the maps of *C*.

The multiplication on *RC* can be understood in several ways, depending on how one presents a free module.

Thinking of the free module as formal linear combinations (which are finite sums), the multiplication is the multiplication (composition) of the category, where defined:

$$\sum a_i f_i \sum b_j g_j = \sum a_i b_j f_i g_j$$

where $f_i g_j = 0$if their composition is not defined. This is defined for any finite sum.

Thinking of the free module as finitely supported functions, the multiplication is defined as a convolution: if $a, b \in RC$ (thought of as functionals on the maps of *C*), then their product is defined as:

$$(a * b)(h) := \sum_{fg=h} a(f)b(g).$$

The latter sum is finite because the functions are finitely supported.

Incidence algebra-style definition

The definition used for incidence algebras assumes that the category *C* is locally finite, is *dual* to the above definition, and defines a *different* object. This isn't a useful assumption for groups, as a group that is locally finite as a category is finite.

A **locally finite category** is one where every map can be written only finitely many ways as a product of non-identity maps. The categorical algebra (in this sense) is defined as above, but allowing all coefficients to be non-zero.

In terms of formal sums, the elements are all formal sums

$$\sum_{f_i \in \mathrm{Hom}(C)} a_i f_i,$$

where there are no restrictions on the a_i (they can all be non-zero).

In terms of functions, the elements are any functions from the maps of *C* to *R*, and multiplication is defined as convolution. The sum in the convolution is always finite because of the local finiteness assumption.

Dual

The module dual of the category algebra (in the group algebra sense of the definition) is the space of all maps from the maps of *C* to *R*, denoted *F(C)*, and has a natural coalgebra structure. Thus for a locally finite category, the dual of a categorical algebra (in the group algebra sense) is the categorical algebra (in the incidence algebra sense), and has both an algebra and coalgebra structure.

Examples

- If *C* is a group (thought of as a groupoid with a single object), then *RC* is the group algebra.
- If *C* is a monoid (thought of as a category with a single object), then *RC* is the semigroup algebra
- If *C* is a partially ordered set, then (using the appropriate definition), *RC* is the incidence algebra.

Generalizations

The above definition does not need the structure of a category, and instead only needs a partial magma. However, this generality is little-studied.

References

- Haigh, John. *On the Möbius Algebra and the Grothendieck Ring of a Finite Category* J. London Math. Soc (2), 21 (1980) 81-92.

External links

- Categorical Algebra [1] at PlanetMath.
- Locally Finite Category [2] at PlanetMath.

References

[1] http://planetmath.org/encyclopedia/AlgebraFormedFromACategory.html
[2] http://planetmath.org/encyclopedia/LocallyFiniteCategory.html

Functor category

In category theory, a branch of mathematics, the functors between two given categories can themselves be turned into a category; the morphisms in this **functor category** are natural transformations between functors. Functor categories are of interest for two main reasons:

- many commonly occurring categories are (disguised) functor categories, so any statement proved for general functor categories is widely applicable;
- every category embeds in a functor category (via the Yoneda embedding); the functor category often has nicer properties than the original category, allowing certain operations that were not available in the original setting.

An element of a functor category is sometimes called a **diagram.**

Definition

Suppose C is a small category (i.e. the objects form a set rather than a proper class) and D is an arbitrary category. The category of functors from C to D, written as Funct(C,D) or D^C, has as objects the covariant functors from C to D, and as morphisms the natural transformations between such functors. Note that natural transformations can be composed: if $\mu(X) : F(X) \to G(X)$ is a natural transformation from the functor $F : C \to D$ to the functor $G : C \to D$, and $\eta(X) : G(X) \to H(X)$ is a natural transformation from the functor G to the functor H, then the collection $\eta(X)\mu(X) : F(X) \to H(X)$ defines a natural transformation from F to H. With this composition of natural transformations (known as vertical composition, see natural transformation), D^C satisfies the axioms of a category.

In a completely analogous way, one can also consider the category of all *contravariant* functors from C to D; we write this as Funct(C^{op},D).

If C and D are both preadditive categories (i.e. their morphism sets are abelian groups and the composition of morphisms is bilinear), then we can consider the category of all additive functors from C to D, denoted by Add(C,D).

Examples

- If I is a small discrete category (i.e. its only morphisms are the identity morphisms), then a functor from I to C essentially consists of a family of objects of C, indexed by I; the functor category C^I can be identified with the corresponding product category: its elements are families of objects in C and its morphisms are families of morphisms in C.
- A directed graph consists of a set of arrows and a set of vertices, and two functions from the arrow set to the vertex set, specifying each arrow's start and end vertex. The category of all directed graphs is thus nothing but the functor category $\mathbf{Set}^C$, where C is the category with two objects connected by two morphisms, and **Set** denotes the category of sets.
- Any group G can be considered as a one-object category in which every morphism is invertible. The category of all G-sets is the same as the functor category $\mathbf{Set}^G$.
- Similar to the previous example, the category of k-linear representations of the group G is the same as the functor category $\boldsymbol{k}\text{-}\mathbf{Vect}^G$ (where ***k*-Vect** denotes the category of all vector spaces over the field k).
- Any ring R can be considered as a one-object preadditive category; the category of left modules over R is the same as the additive functor category Add(R,**Ab**) (where **Ab** denotes the category of abelian groups), and the category of right R-modules is Add(R^{op},**Ab**). Because of this example, for any preadditive category C, the category Add(C,**Ab**) is sometimes called the "category of left modules over C" and Add(C^{op},**Ab**) is the category of right modules over C.
- The category of presheaves on a topological space X is a functor category: we turn the topological space into a category C having the open sets in X as objects and a single morphism from U to V if and only if U is contained in V. The category of presheaves of sets (abelian groups, rings) on X is then the same as the category of

contravariant functors from C to **Set** (or **Ab** or **Ring**). Because of this example, the category Funct(C^{op}, **Set**) is sometimes called the "category of presheaves of sets on C" even for general categories C not arising from a topological space. To define sheaves on a general category C, one needs more structure: a Grothendieck topology on C. (Some authors refer to categories that are equivalent to $\mathbf{Set}^C$ as *presheaf categories*.[1])

Facts

Most constructions that can be carried out in D can also be carried out in D^C by performing them "componentwise", separately for each object in C. For instance, if any two objects X and Y in D have a product $X \times Y$, then any two functors F and G in D^C have a product $F \times G$, defined by $(F \times G)(c) = F(c) \times G(c)$ for every object c in C. Similarly, if $\eta_c : F(c) \to G(c)$ is a natural transformation and each η_c has a kernel K_c in the category D, then the kernel of η in the functor category D^C is the functor K with $K(c) = K_c$ for every object c in C.

As a consequence we have the general rule of thumb that the functor category D^C shares most of the "nice" properties of D:

- if D is complete (or cocomplete), then so is D^C;
- if D is an abelian category, then so is D^C;

We also have:

- if C is any small category, then the category $\mathbf{Set}^C$ of presheaves is a topos.

So from the above examples, we can conclude right away that the categories of directed graphs, G-sets and presheaves on a topological space are all complete and cocomplete topoi, and that the categories of representations of G, modules over the ring R, and presheaves of abelian groups on a topological space X are all abelian, complete and cocomplete.

The embedding of the category C in a functor category that was mentioned earlier uses the Yoneda lemma as its main tool. For every object X of C, let Hom(-,X) be the contravariant representable functor from C to **Set**. The Yoneda lemma states that the assignment

$$X \mapsto \mathrm{Hom}(-, X)$$

is a full embedding of the category C into the category Funct(C^{op},**Set**). So C naturally sits inside a topos.

The same can be carried out for any preadditive category C: Yoneda then yields a full embedding of C into the functor category Add(C^{op},**Ab**). So C naturally sits inside an abelian category.

The intuition mentioned above (that constructions that can be carried out in D can be "lifted" to D^C) can be made precise in several ways; the most succinct formulation uses the language of adjoint functors. Every functor $F : D \to E$ induces a functor $F^C : D^C \to E^C$ (by composition with F). If F and G is a pair of adjoint functors, then F^C and G^C is also a pair of adjoint functors.

The functor category D^C has all the formal properties of an exponential object; in particular the functors from $E \times C \to D$ stand in a natural one-to-one correspondence with the functors from E to D^C. The category **Cat** of all small categories with functors as morphisms is therefore a cartesian closed category.

See also

- Diagram (category theory)

References

[1] Tom Leinster (2004). *Higher Operads, Higher Categories* (http://www.maths.gla.ac.uk/~tl/book.html). Cambridge University Press. .

Higher category theory

Higher category theory is the part of category theory at a *higher order*, which means that some equalities are replaced by explicit arrows in order to be able to explicitly study the structure behind those equalities.

Strict higher categories

N-categories are defined inductively using the enriched category theory: 0-categories are sets, and (n+1)-categories are categories enriched over the monoidal category of n-categories (with the monoidal structure given by finite products).[1] This construction is well defined, as shown in the article on n-categories. This concept introduces higher arrows, higher compositions and higher identities, which must well behave together. For example, the category of small categories is in fact a 2-category, with natural transformations as second degree arrows. However this concept is too strict for some purposes (for example, homotopy theory), where "weak" structures arise in the form of higher categories.[2]

Weak higher categories

In weak n-categories, the associativity and identity conditions are no longer strict (that is, they are not given by equalities), but rather are satisfied up to an isomorphism of the next level. An example in topology is the composition of paths, which is associative only up to homotopy. These isomorphisms must well behave between them and expressing this is the difficulty in the definition of weak n-categories. Weak 2-categories, also called bicategories, were the first to be defined explicitly. A particularity of these is that a bicategory with one object is exactly a monoidal category, so that bicategories can be said to be "monoidal categories with many objects." Weak 3-categories, also called tricategories, and higher-level generalizations are increasingly harder to define explicitly. Several definitions have been given, and telling when they are equivalent, and in what sense, has become a new object of study in category theory.

References

[1] Leinster, pp 18-19
[2] Baez, p 6

- John C. Baez; James Dolan (1998). *Categorification* (http://arxiv.org/abs/math/9802029).
- Tom Leinster (2004). *Higher Operads, Higher Categories* (http://www.maths.gla.ac.uk/~tl/book.html). Cambridge University Press.

See also

- Network Science
- Polytely
- Higher-dimensional algebra

External links

- John Baez Tale of *n*-Categories (http://math.ucr.edu/home/baez/week73.html)

Categorical group

In mathematics, a **2-group**, or **2-dimensional higher group**, is a certain combination of group and groupoid. The 2-groups are part of a larger hierarchy of *n*-groups. In some of the literature, 2-groups are also called **gr-categories** or **groupal groupoids**.

Definition

A 2-group is a monoidal category G in which every morphism is invertible and every object has a weak inverse. (Here, a *weak inverse* of an object x is an object y such that xy and yx are both isomorphic to the unit object.)

Strict 2-groups

Much of the literature focuses on *strict 2-groups*. A strict 2-group is a *strict* monoidal category in which every morphism is invertible and every object has a strict inverse (so that xy and yx are actually equal to the unit object).

A strict 2-group is a group object in a category of categories; as such, they are also called *groupal categories*. Conversely, a strict 2-group is a category object in the category of groups; as such, they are also called *categorical groups*. They can also be identified with crossed modules, and are most often studied in that form. Thus, 2-groups in general can be seen as a weakening of crossed modules.

Every 2-group is equivalent to a strict 2-group, although this can't be done coherently: it doesn't extend to 2-group homomorphisms.

Properties

Weak inverses can always be assigned coherently: one can define a functor on any 2-group G that assigns a weak inverse to each object and makes that object an adjoint equivalance in the monoidal category G.

Given a bicategory B and an object x of B, there is an *automorphism 2-group* of x in B, written $\mathrm{Aut}_B(x)$. The objects are the automorphisms of x, with multiplication given by composition, and the morphisms are the invertible 2-morphisms between these. If B is a 2-groupoid (so all objects and morphisms are weakly invertible) and x is its only object, then $\mathrm{Aut}_B(x)$ is the only data left in B. Thus, 2-groups may be identified with one-object 2-groupoids, much as groups may be idenitified with one-object groupoids and monoidal categories may be identified with one-object bicategories.

If G is a strict 2-group, then the objects of G form a group, called the *underlying group* of G and written G_0. This will not work for arbitrary 2-groups; however, if one identifies isomorphic objects, then the equivalence classes form a group, called the *fundamental group* of G and written $\pi_1(G)$. (Note that even for a strict 2-group, the fundamental group will only be a quotient group of the underlying group.)

As a monoidal category, any 2-group G has a unit object I_G. The automorphism group of I_G is an abelian group by the Eckmann–Hilton argument, written $\mathrm{Aut}(I_G)$ or $\pi_2(G)$.

The fundamental group of G acts on either side of $\pi_2(G)$, and the associator of G (as a monoidal category) defines an element of the cohomology group $\mathrm{H}^3(\pi_1(G),\pi_2(G))$. In fact, 2-groups are classified in this way: given a group π_1, an abelian group π_2, a group action of π_1 on π_2, and an element of $\mathrm{H}^3(\pi_1,\pi_2)$, there is a unique (up to equivalence) 2-group G with $\pi_1(G)$ isomorphic to π_1, $\pi_2(G)$ isomorphic to π_2, and the other data corresponding.

The fundamental 2-group

Given a topological space X and a point x in that space, there is a *fundamental 2-group* of X at x, written $\Pi_2(X,x)$. As a monoidal category, the objects are loops at x, with multiplication given by concatenation, and the morphisms are basepoint-preserving homotopies between loops, with these morphisms identified if they are themselves homotopic.

Conversely, given any 2-group G, one can find a unique (up to weak homotopy equivalence) pointed connected space whose fundamental 2-group is G and whose homotopy groups π_n are trivial for $n > 2$. In this way, 2-groups classify pointed connected weak homotopy 2-types. This is a generalisation of the construction of Eilenberg–Mac Lane spaces.

If X is a topological space with basepoint x, then the fundamental group of X at x is the same as the fundamental group of the fundamental 2-group of X at x; that is,

$$\pi_1(X,x) = \pi_1(\Pi_2(X,x)).$$

This fact is the origin of the term "fundamental" in both of its 2-group instances.

Similarly,

$$\pi_2(X,x) = \pi_2(\Pi_2(X,x)).$$

Thus, both the first and second homotopy groups of a space are contained within its fundamental 2-group. As this 2-group also defines an action of $\pi_1(X,x)$ on $\pi_2(X,x)$ and an element of the cohomology group $H^3(\pi_1(X,x),\pi_2(X,x))$, this is precisely the data needed to form the Postnikov tower of X if X is a pointed connected homotopy 2-type.

References

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- John C. Baez and Danny Stevenson, The Classifying Space of a Topological 2-Group [2].
- Hendryk Pfeiffer, 2-Groups, trialgebras and their Hopf categories of representations [3], Adv. Math. 212 No. 1 (2007) 62–108.
- 2-group [4] at the *n*-Category Lab.

External links

- 2008 Workshop on Categorical Groups [5] at the Centre de Recerca Matemàtica

2-category

In category theory, a 2-category is a category with "morphisms between morphisms"; that is, where each hom set itself carries the structure of a category. It can be formally defined as a category enriched over **Cat** (the category of categories and functors, with the monoidal structure given by product of categories).

Definition

A 2-category **C** consists of:

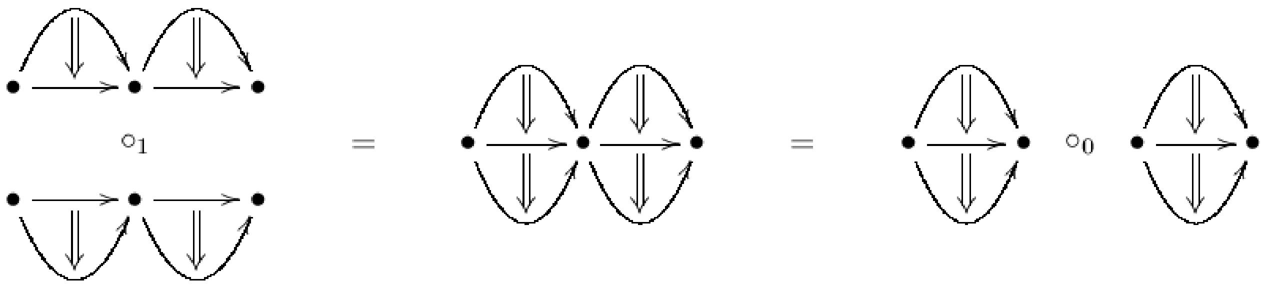
- A class of *0-cells* (or *objects*) *A*, *B*, ...
- For all objects *A* and *B*, a category $\mathbf{C}(A,B)$. The objects $f : A \to B$ of this category are called *1-cells* and its morphisms $\alpha : f \Rightarrow g$ are called *2-cells*; the composition in this category is usually written $\circ$ or $\circ_1$ and called *vertical composition* or *composition along a 1-cell*.
- For any object *A* there is a functor from the terminal category (with one object and one arrow) to $\mathbf{C}(A,A)$, that picks out the identity 1-cell id_A on *A* and its identity 2-cell $\mathrm{id}_{\mathrm{id}_A}$. In practice these two are often denoted simply by *A*.
- For all objects *A*, *B* and *C*, there is a functor $\circ_0 : \mathbf{C}(B,C) \times \mathbf{C}(A,B) \to \mathbf{C}(A,C)$, called *horizontal composition* or *composition along a 0-cell*, which is associative and admits the identity 2-cells of id_A as identities. The composition symbol $\circ_0$ is often omitted, the horizontal composite of 2-cells $\alpha : f \Rightarrow g : A \to B$ and $\beta : f' \Rightarrow g' : B \to C$ being written simply as $\beta\alpha : f'f \Rightarrow g'g : A \to C$.

The notion of 2-category differs from the more general notion of a bicategory in that composition of (1-)morphisms is required to be strictly associative, whereas in a bicategory it need only be associative up to a 2-isomorphism. The axioms of a 2-category are consequences of their definition as **Cat**-enriched categories:

- Vertical composition is associative and unital, the units being the identity 2-cells id f.
- Horizontal composition is also (strictly) associative and unital, the units being the identity 2-cells $A = \mathrm{id}_{\mathrm{id}} A$ on the identity 1-cells id A.
- The interchange law holds; i.e. it is true that for composable 2-cells $\alpha, \beta, \gamma, \delta$

  $$(\alpha \circ_0 \beta) \circ_1 (\gamma \circ_0 \delta) = (\alpha \circ_1 \gamma) \circ_0 (\beta \circ_1 \delta)$$

The interchange law follows from the fact that $\circ_0$ is a functor between hom categories. It can be drawn as a pasting diagram as follows:



Here the left-hand diagram denotes the vertical composition of horizontal composites, the right-hand diagram denotes the horizontal composition of vertical composites, and the diagram in the centre is the customary representation of both.

Doctrines

In mathematics, a **doctrine** is simply a 2-category which is heuristically regarded as a system of theories. For example, algebraic theories, as invented by Lawvere, is an example of a doctrine, as are multi-sorted theories, operads, categories, and toposes.

The objects of the 2-category are called *theories*, the 1-morphisms $f\colon A \to B$ are called *models* of the A in B, and the 2-morphisms are called *morphisms between models.*

The distinction between a 2-category and a doctrine is really only heuristic: one does not typically consider a 2-category to be populated by theories as objects and models as morphisms. It is this vocabulary that makes the theory of doctrines worth while.

For example, the 2-category **Cat** of categories, functors, and natural transformations is a doctrine. One sees immediately that all presheaf categories are categories of models.

As another example, one may take the subcategory of **Cat** consisting only of product-preserving functors as 1-morphisms. This is the doctrine of multi-sorted algebraic theories. If one only wanted 1-sorted algebraic theories, one would restrict the objects to only those categories that are generated under products by a single object.

Doctrines were invented by J. M. Beck.

See also

- n-category

References

- *Generalised algebraic models*, by Claudia Centazzo.

N-category

In mathematics, ***n*-categories** are a high-order generalization of the notion of category. The category of (small) *n*-categories *n*-**Cat** is defined by induction on *n* by:

- the category 0-**Cat** is the category **Set** of sets and functions,
- the category (*n*+1)-**Cat** is the category of categories enriched over the category *n*-**Cat**.

An ***n*-category** is then an object of *n*-**Cat**.

The monoidal structure of **Set** is the one given by the cartesian product as tensor and a singleton as unit. In fact any category with finite products can be given a monoidal structure. The recursive construction of *n*-**Cat** works fine because if a category C has finite products, the category of C-enriched categories has finite products too.

In particular, the category 1-**Cat** is the category **Cat** of small categories and functors.

n-categories have given rise to higher category theory, where several types of *n*-categories are studied. The necessity of weakening the definition of an *n*-category for homotopic purposes has led to the definition of weak *n*-categories. For distinction, the *n*-categories as defined above are called strict.

See also

- 2-category
- n-category number
- Weak n-category

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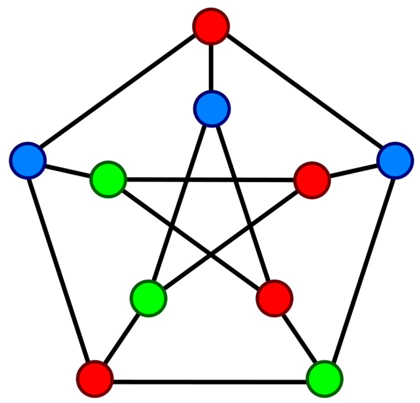
Colored graph



A proper vertex coloring of the Petersen graph with 3 colors, the minimum number possible.

In graph theory, **graph coloring** is a special case of graph labeling; it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a **vertex coloring**. Similarly, an **edge coloring** assigns a color to each edge so that no two adjacent edges share the same color, and a **face coloring** of a planar graph assigns a color to each face or region so that no two faces that share a boundary have the same color.

Vertex coloring is the starting point of the subject, and other coloring problems can be transformed into a vertex version. For example, an edge coloring of a graph is just a vertex coloring of its line graph, and a face coloring of a planar graph is just a vertex coloring of its planar dual. However, non-vertex coloring problems are often stated and studied *as is*. That is partly for perspective, and partly because some problems are best studied in non-vertex form, as for instance is edge coloring.

The convention of using colors originates from coloring the countries of a map, where each face is literally colored. This was generalized to coloring the faces of a graph embedded in the plane. By planar duality it became coloring the vertices, and in this form it generalizes to all graphs. In mathematical and computer representations it is typical to use the first few positive or nonnegative integers as the "colors". In general one can use any finite set as the "color set". The nature of the coloring problem depends on the number of colors but not on what they are.

Graph coloring enjoys many practical applications as well as theoretical challenges. Beside the classical types of problems, different limitations can also be set on the graph, or on the way a color is assigned, or even on the color itself. It has even reached popularity with the general public in the form of the popular number puzzle Sudoku. Graph coloring is still a very active field of research.

Note: Many terms used in this article are defined in Glossary of graph theory.

History

The first results about graph coloring deal almost exclusively with planar graphs in the form of the coloring of *maps*. While trying to color a map of the counties of England, Francis Guthrie postulated the four color conjecture, noting that four colors were sufficient to color the map so that no regions sharing a common border received the same color. Guthrie's brother passed on the question to his mathematics teacher Augustus de Morgan at University College, who mentioned it in a letter to William Hamilton in 1852. Arthur Cayley raised the problem at a meeting of the London Mathematical Society in 1879. The same year, Alfred Kempe published a paper that claimed to establish the result, and for a decade the four color problem was considered solved. For his accomplishment Kempe was elected a Fellow of the Royal Society and later President of the London Mathematical Society.[1]

In 1890, Heawood pointed out that Kempe's argument was wrong. However, in that paper he proved the five color theorem, saying that every planar map can be colored with no more than *five* colors, using ideas of Kempe. In the following century, a vast amount of work and theories were developed to reduce the number of colors to four, until the four color theorem was finally proved in 1976 by Kenneth Appel and Wolfgang Haken. Perhaps surprisingly, the proof went back to the ideas of Heawood and Kempe and largely disregarded the intervening developments.[2] The proof of the four color theorem is also noteworthy for being the first major computer-aided proof.

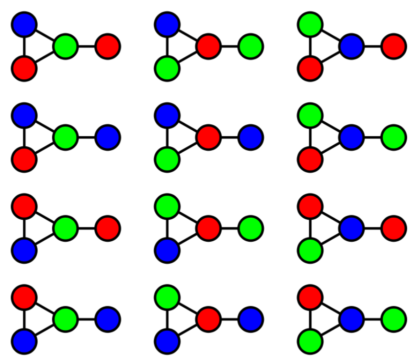
In 1912, George David Birkhoff introduced the chromatic polynomial to study the coloring problems, which was generalised to the Tutte polynomial by Tutte, important structures in algebraic graph theory. Kempe had already drawn attention to the general, non-planar case in 1879,[3] and many results on generalisations of planar graph coloring to surfaces of higher order followed in the early 20th century.

In 1960, Claude Berge formulated another conjecture about graph coloring, the *strong perfect graph conjecture*, originally motivated by an information-theoretic concept called the zero-error capacity of a graph introduced by Shannon. The conjecture remained unresolved for 40 years, until it was established as the celebrated strong perfect graph theorem in 2002 by Chudnovsky, Robertson, Seymour, Thomas 2002.

Graph coloring has been studied as an algorithmic problem since the early 1970s: the chromatic number problem is one of Karp's 21 NP-complete problems from 1972, and at approximately the same time various exponential-time algorithms were developed based on backtracking and on the deletion-contraction recurrence of Zykov (1949). One of the major applications of graph coloring, register allocation in compilers, was introduced in 1981.

Definition and terminology

Vertex coloring


This graph can be 3-colored in 12 different ways.

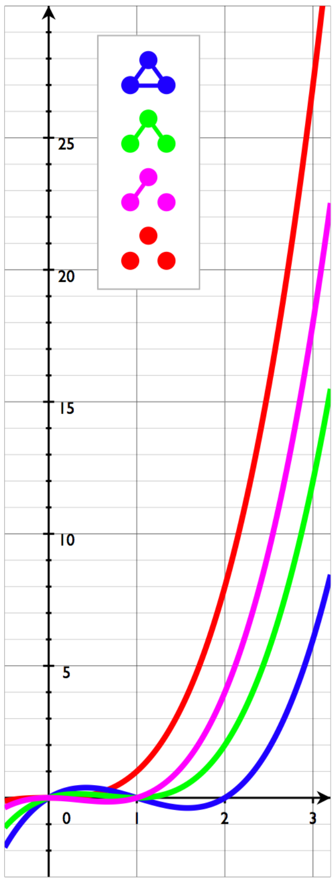
When used without any qualification, a **coloring** of a graph is almost always a *proper vertex coloring*, namely a labelling of the graph's vertices with colors such that no two vertices sharing the same edge have the same color. Since a vertex with a loop could never be properly colored, it is understood that graphs in this context are loopless.

The terminology of using *colors* for vertex labels goes back to map coloring. Labels like *red* and *blue* are only used when the number of colors is small, and normally it is understood that the labels are drawn from the integers {1,2,3,...}.

A coloring using at most k colors is called a (proper) ***k*-coloring**. The smallest number of colors needed to color a graph G is called its **chromatic number**, $\chi(G)$. A graph that can be assigned a (proper) k-coloring is ***k*-colorable**, and it is ***k*-chromatic** if its chromatic number is exactly k. A subset of vertices assigned to the same color is called a *color class*, every such class forms an independent set. Thus, a k-coloring is the same as a partition of the vertex set into k independent sets, and the terms *k-partite* and *k-colorable* have the same meaning.

Chromatic polynomial

The **chromatic polynomial** counts the number of ways a graph can be colored using no more than a given number of colors. For example, using three colors, the graph in the image to the right can be colored in 12 ways. With only two colors, it cannot be colored at all. With four colors, it can be colored in $24 + 4 \cdot 12 = 72$ ways: using all four colors, there are $4! = 24$ valid colorings (*every* assignment of four colors to *any* 4-vertex graph is a proper coloring); and for every choice of three of the four colors, there are 12 valid 3-colorings. So, for the graph in the example, a table of the number of valid colorings would start like this:



All nonisomorphic graphs on 3 vertices and their chromatic polynomials. The empty graph E_3 (red) admits a 1-coloring, the others admit no such colorings. The green graph admits 12 colorings with 3 colors.

Available colors	1	2	3	4	…
Number of colorings	0	0	12	72	…

The chromatic polynomial is a function $P(G, t)$ that counts the number of t-colorings of G. As the name indicates, for a given G the function is indeed a polynomial in t. For the example graph, $P(G, t) = t(t - 1)^2(t - 2)$, and indeed $P(G, 4) = 72$.

The chromatic polynomial includes at least as much information about the colorability of G as does the chromatic number. Indeed, χ is the smallest positive integer that is not a root of the chromatic polynomial

$$\chi(G) = \min\{k \colon P(G, k) > 0\}.$$

Chromatic polynomials for certain graphs

Triangle K_3	$t(t-1)(t-2)$
Complete graph K_n	$t(t-1)(t-2)\cdots(t-(n-1))$
Tree with n vertices	$t(t-1)^{n-1}$
Cycle C_n	$(t-1)^n+(-1)^n(t-1)$
Petersen graph	$t(t-1)(t-2)(t^7-12t^6+67t^5-230t^4+529t^3-814t^2+775t-352)$

Edge coloring

An **edge coloring** of a graph, is a proper coloring of the *edges*, meaning an assignment of colors to edges so that no vertex is incident to two edges of the same color. An edge coloring with k colors is called a k-edge-coloring and is equivalent to the problem of partitioning the edge set into k matchings. The smallest number of colors needed for an edge coloring of a graph G is the **chromatic index**, or **edge chromatic number**, $\chi'(G)$. A **Tait coloring** is a 3-edge coloring of a cubic graph. The four color theorem is equivalent to the assertion that every planar cubic bridgeless graph admits a Tait coloring.

Properties

Bounds on the chromatic number

Assigning distinct colors to distinct vertices always yields a proper coloring, so

$$1 \le \chi(G) \le n.$$

The only graphs that can be 1-colored are edgeless graphs, and the complete graph K_n of n vertices requires $\chi(K_n)=n$ colors. In an optimal coloring there must be at least one of the graph's m edges between every pair of color classes, so

$$\chi(G)(\chi(G)-1) \le 2m.$$

If G contains a clique of size k, then at least k colors are needed to color that clique; in other words, the chromatic number is at least the clique number:

$$\chi(G) \ge \omega(G).$$

For interval graphs this bound is tight.

The 2-colorable graphs are exactly the bipartite graphs, including trees and forests. By the four color theorem, every planar graph can be 4-colored.

A greedy coloring shows that every graph can be colored with one more color than the maximum vertex degree,

$$\chi(G) \le \Delta(G)+1.$$

Complete graphs have $\chi(G)=n$ and $\Delta(G)=n-1$, and odd cycles have $\chi(G)=3$ and $\Delta(G)=2$, so for these graphs this bound is best possible. In all other cases, the bound can be slightly improved; Brooks' theorem[4] states that

> **Brooks' theorem:** $\chi(G) \le \Delta(G)$ for a connected, simple graph G, unless G is a complete graph or an odd cycle.

Graphs with high chromatic number

Graphs with large cliques have high chromatic number, but the opposite is not true. The Grötzsch graph is an example of a 4-chromatic graph without a triangle, and the example can be generalised to the Mycielskians.

Mycielski's Theorem: There exist triangle-free graphs with arbitrarily high chromatic number.

From Brooks's theorem, graphs with high chromatic number must have high maximum degree. Another local property that leads to high chromatic number is the presence of a large clique. But colorability is not an entirely local phenomenon: A graph with high girth looks locally like a tree, because all cycles are long, but its chromatic number need not be 2:

Theorem (Erdős): There exist graphs of arbitrarily high girth and chromatic number.

Bounds on the chromatic index

An edge coloring of G is a vertex coloring of its line graph $L(G)$, and vice versa. Thus,

$$\chi(G) = \chi'(L(G)).$$

There is a strong relationship between edge colorability and the graph's maximum degree $\Delta(G)$. Since all edges incident to the same vertex need their own color, we have

$$\chi'(G) \geq \Delta(G).$$

Moreover,

König's theorem: $\chi'(G) = \Delta(G)$ if G is bipartite.

In general, the relationship is even stronger than what Brooks's theorem gives for vertex coloring:

Vizing's Theorem: A graph of maximal degree $\triangle$ has edge-chromatic number $\triangle$ or $\Delta + 1$.

Other properties

For planar graphs, vertex colorings are essentially dual to nowhere-zero flows.

About infinite graphs, much less is known. The following is one of the few results about infinite graph coloring:

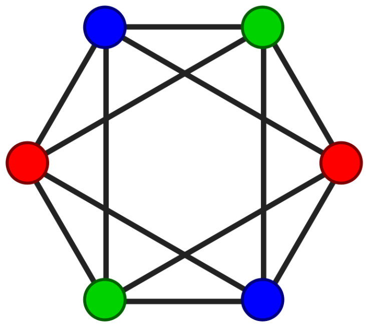
If all finite subgraphs of an infinite graph *G* are *k*-colorable, then so is *G*, under the assumption of the axiom of choice (de Bruijn & Erdős 1951).

Open problems

The chromatic number of the plane, where two points are adjacent if they have unit distance, is unknown, although it is one of 4, 5, 6, or 7. Other open problems concerning the chromatic number of graphs include the Hadwiger conjecture stating that every graph with chromatic number *k* has a complete graph on *k* vertices as a minor, the Erdős–Faber–Lovász conjecture bounding the chromatic number of unions of complete graphs that have at exactly one vertex in common to each pair, and the Albertson conjecture that among *k*-chromatic graphs the complete graphs are the ones with smallest crossing number.

When Birkhoff and Lewis introduced the chromatic polynomial in their attack on the four-color theorem, they conjectured that for planar graphs G, the polymomial $P(G, t)$ has no zeros in the region $[4, \infty)$. Although it is known that such a chromatic polynomial has no zeros in the region $[5, \infty)$ and that $P(G, 4) \neq 0$, their conjecture is still unresolved. It also remains an unsolved problem to characterize graphs which have the same chromatic polynomial and to determine which polynomials are chromatic.

Algorithms

Graph coloring	
	
Decision	
Name	Graph coloring, vertex coloring, k-coloring
Input	Graph G with n vertices. Integer k
Output	Does G admit a proper vertex coloring with k colors?
Running time	$O(2^n n)$
Complexity	NP-complete
Reduction from	3-Satisfiability
Garey–Johnson	GT4
Optimisation	
Name	Chromatic number
Input	Graph G with n vertices.
Output	$\chi(G)$
Complexity	NP-hard
Approximability	$O(n (\log n)^{-3}(\log \log n)^2)$
Inapproximability	$O(n^{1-\varepsilon})$ unless P = NP
Counting problem	
Name	Chromatic polymomial
Input	Graph G with n vertices. Integer k
Output	The number $P(G,k)$ of proper k-colorings of G
Running time	$O(2^n n)$
Complexity	#P-complete
Approximability	FPRAS for restricted cases
Inapproximability	No PTAS unless P = NP

Efficient algorithms

Determining if a graph can be colored with 2 colors is equivalent to determining whether or not the graph is bipartite, and thus computable in linear time using breadth-first search. More generally, the chromatic number and a corresponding coloring of perfect graphs can be computed in polynomial time using semidefinite programming. Closed formulas for chromatic polynomial are known for many classes of graphs, such as forest, chordal graphs, cycles, wheels, and ladders, so these can be evaluated in polynomial time.

Brute-force search

Brute-force search for a *k*-coloring considers every of the k^n assignments of *k* colors to *n* vertices and checks for each if it is legal. To compute the chromatic number and the chromatic polynomial, this procedure is used for every $k = 1, \ldots, n-1$, impractical for all but the smallest input graphs.

Contraction

The contraction G/uv of graph G is the graph obtained by identifying the vertices u and v, removing any edges between them, and replacing them with a single vertex w where any edges that were incident on u or v are redirected to w. This operation plays a major role in the analysis of graph coloring.
The chromatic number satisfies the recurrence relation:

$$\chi(G) = \min\{\chi(G+uv), \chi(G/uv)\}$$

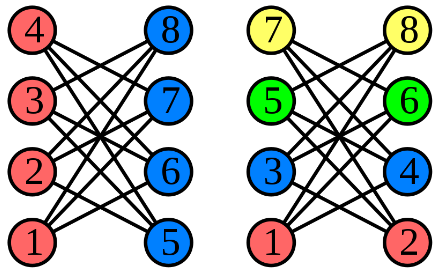
due to Zykov (1949), where u and v are nonadjacent vertices, $G+uv$ is the graph with the edge uv added. Several algorithms are based on evaluating this recurrence, the resulting computation tree is sometimes called a Zykov tree. The running time is based on the heuristic for choosing the vertices u and v.

The chromatic polynomial satisfies following recurrence relation

$$P(G-uv, k) = P(G/uv, k) + P(G, k)$$

where u and v are adjacent vertices and $G-uv$ is the graph with the edge uv removed. $P(G-uv, k)$ represents the number of possible proper colorings of the graph, when the vertices may have same or different colors. The number of proper colorings therefore come from the sum of two graphs. If the vertices u and v have different colors, then we can as well consider a graph, where u and v are adjacent. If u and v have the same colors, we may as well consider a graph, where u and v are contracted. Tutte's curiosity about which other graph properties satisfied this recurrence led him to discover a bivariate generalization of the chromatic polynomial, the Tutte polynomial.
The expressions give rise to a recursive procedure, called the *deletion–contraction algorithm*, which forms the basis of many algorithms for graph coloring. The running time satisfies the same recurrence relation as the Fibonacci numbers, so in the worst case, the algorithm runs in time within a polynomial factor of $((1+\sqrt{5})/2)^{n+m} = O(1.6180)^{n+m}$.[5] The analysis can be improved to within a polynomial factor of the number $t(G)$ of spanning trees of the input graph.[6] In practice, branch and bound strategies and graph isomorphism rejection are employed to avoid some recursive calls, the running time depends on the heuristic used to pick the vertex pair.

Greedy coloring



Two greedy colorings of the same graph using different vertex orders. The right example generalises to 2-colorable graphs with n vertices, where the greedy algorithm expends $n/2$ colors.

The greedy algorithm considers the vertices in a specific order v_1,…, v_n and assigns to v_i the smallest available color not used by v_i's neighbours among v_1,…, v_{i-1}, adding a fresh color if needed. The quality of the resulting coloring depends on the chosen ordering. There exists an ordering that leads to a greedy coloring with the optimal number of $\chi(G)$ colors. On the other hand, greedy colorings can be arbitrarily bad; for example, the crown graph on n vertices can be 2-colored, but has an ordering that leads to a greedy coloring with $n/2$ colors.

If the vertices are ordered according to their degrees, the resulting greedy coloring uses at most $\max_i \min\{d(x_i)+1, i\}$ colors, at most one more than the graph's maximum degree. This heuristic is sometimes called the Welsh–Powell algorithm.[7] Another heuristic establishes the ordering dynamically while the algorithm proceeds, choosing next the vertex adjacent to the largest number of different colors.[8] Many other graph coloring heuristics are similarly based on greedy coloring for a specific static or dynamic strategy of ordering the vertices, these algorithms are sometimes called **sequential coloring** algorithms.

Computational complexity

Graph coloring is computationally hard. It is NP-complete to decide if a given graph admits a k-coloring for a given k except for the cases $k = 1$ and $k = 2$. Especially, it is NP-hard to compute the chromatic number. Graph coloring remains NP-complete even on planar graphs of degree 4.[9]

The best known approximation algorithm computes a coloring of size at most within a factor $O(n(\log n)^{-3}(\log\log n)^2)$ of the chromatic number.[10] For all $\varepsilon > 0$, approximating the chromatic number within $n^{1-\varepsilon}$ is NP-hard.[11]

It is also NP-hard to color a 3-colorable graph with 4 colors[12] and a k-colorable graph with $k^{(\log k)/25}$ colors for sufficiently large constant k.[13]

Computing the coefficients of the chromatic polynomial is #P-hard. In fact, even computing the value of $\chi(G,k)$ is #P-hard at any rational point k except for $k = 1$ and $k = 2$.[14] There is no FPRAS for evaluating the chromatic polynomial at any rational point $k \geq 1.5$ except for $k = 2$ unless NP = RP.[15]

For edge coloring, the proof of Vizing's result gives an algorithm that uses at most $\Delta+1$ colors. However, deciding between the two candidate values for the edge chromatic number is NP-complete.[16] In terms of approximation algorithms, Vizing's algorithm shows that the edge chromatic number can be approximated within 4/3, and the hardness result shows that no $(4/3 - \varepsilon)$-algorithm exists for any $\varepsilon > 0$ unless P = NP. These are among the oldest results in the literature of approximation algorithms, even though neither paper makes explicit use of that notion.[17]

Parallel and distributed algorithms

In the field of distributed algorithms, graph coloring is closely related to the problem of *symmetry breaking*. In a symmetric graph, a deterministic distributed algorithm cannot find a proper vertex coloring. Some auxiliary information is needed in order to break symmetry. A standard assumption is that initially each node has a *unique identifier*, for example, from the set {1, 2, ..., n} where n is the number of nodes in the graph. Put otherwise, we assume that we are given an n-coloring. The challenge is to *reduce* the number of colors from n to, e.g., $\Delta + 1$.

A straightforward distributed version of the greedy algorithm for $(\Delta + 1)$-coloring requires $\Theta(n)$ communication rounds in the worst case − information may need to be propagated from one side of the network to another side. However, much faster algorithms exist, at least if the maximum degree Δ is small.

The simplest interesting case is an n-cycle. Richard Cole and Uzi Vishkin[18] show that there is a distributed algorithm that reduces the number of colors from n to $O(\log n)$ in one synchronous communication step. By iterating the same procedure, it is possible to obtain a 3-coloring of an n-cycle in $O(\log^* n)$ communication steps (assuming that we have unique node identifiers).

The function log*, iterated logarithm, is an extremely slowly growing function, "almost constant". Hence the result by Cole and Vishkin raised the question of whether there is a *constant-time* distribute algorithm for 3-coloring an n-cycle. Linial (1992) showed that this is not possible: any deterministic distributed algorithm requires $\Omega(\log^* n)$ communication steps to reduce an n-coloring to a 3-coloring in an n-cycle.

The technique by Cole and Vishkin can be applied in arbitrary bounded-degree graphs as well; the running time is $\mathrm{poly}(\Delta) + O(\log^* n)$.[19] The current fastest known algorithm for $(\Delta + 1)$-coloring is due to Leonid Barenboim and Michael Elkin, which runs in time $O(\Delta) + \log^*(n)/2$,[20] which is optimal in terms of n since the constant factor 1/2 cannot be improved due to Linial's lower bound.

The problem of edge coloring has also been studied in the distributed model. Panconesi & Rizzi (2001) achieve a $(2\Delta - 1)$-coloring in $O(\Delta + \log^* n)$ time in this model. The lower bound lower bound for distributed vertex coloring due to Linial (1992) applies to the distributed edge coloring problem as well.

Applications

Scheduling

Vertex coloring models to a number of scheduling problems.[21] In the cleanest form, a given set of jobs need to be assigned to time slots, each job requires one such slot. Jobs can be scheduled in any order, but pairs of jobs may be in *conflict* in the sense that they may not be assigned to the same time slot, for example because they both rely on a shared resource. The corresponding graph contains a vertex for every job and an edge for every conflicting pair of jobs. The chromatic number of the graph is exactly the minimum *makespan*, the optimal time to finish all jobs without conflicts.

Details of the scheduling problem define the structure of the graph. For example, when assigning aircrafts to flights, the resulting conflict graph is an interval graph, so the coloring problem can be solved efficiently. In bandwidth allocation to radio stations, the resulting conflict graph is a unit disk graph, so the coloring problem is 3-approximable.

Register allocation

A compiler is a computer program that translates one computer language into another. To improve the execution time of the resulting code, one of the techniques of compiler optimization is register allocation, where the most frequently used values of the compiled program are kept in the fast processor registers. Ideally, values are assigned to registers so that they can all reside in the registers when they are used.

The textbook approach to this problem is to model it as a graph coloring problem.[22] The compiler constructs an *interference graph*, where vertices are symbolic registers and an edge connects two nodes if they are needed at the same time. If the graph can be colored with k colors then the variables can be stored in k registers.

Other applications

The problem of coloring a graph has found a number of applications, including pattern matching.

The recreational puzzle Sudoku can be seen as completing a 9-coloring on given specific graph with 81 vertices.

Other colorings

Ramsey theory

An important class of *improper* coloring problems is studied in Ramsey theory, where the graph's edges are assigned to colors, and there is no restriction on the colors of incident edges. A simple example is the friendship theorem says that in any coloring of the edges of K_6 the complete graph of six vertices there will be a monochromatic triangle; often illustrated by saying that any group of six people either has three mutual strangers or three mutual acquaintances. Ramsey theory is concerned with generalisations of this idea to seek regularity amid disorder, finding general conditions for the existence of monochromatic subgraphs with given structure.

Other colorings

List coloring
: Each vertex chooses from a list of colors

List edge-coloring
: Each edge chooses from a list of colors

Total coloring
: Vertices and edges are colored

Harmonious coloring
: Every pair of colors appears on at most one edge

Complete coloring
: Every pair of colors appears on at least one edge

Exact coloring
: Every pair of colors appears on exactly one edge

Acyclic coloring
: Every 2-chromatic subgraph is acyclic

Star coloring
: Every 2-chromatic subgraph is a disjoint collection of stars

Strong coloring
: Every color appears in every partition of equal size exactly once

Strong edge coloring
: Edges are colored such that each color class induces a matching (equivalent to coloring the square of the line graph)

Equitable coloring
: The sizes of color classes differ by at most one

T-coloring
: Distance between two colors of adjacent vertices must not belong to fixed set T

Rank coloring
: If two vertices have the same color i, then every path between them contain a vertex with color greater than i

Interval edge-coloring
: A color of edges meeting in a common vertex must be contiguous

Circular coloring
: Motivated by task systems in which production proceeds in a cyclic way

Path coloring
: Models a routing problem in graphs

Fractional coloring
: Vertices may have multiple colors, and on each edge the sum of the color parts of each vertex is not greater than one

Oriented coloring
: Takes into account orientation of edges of the graph

Cocoloring
: An improper vertex coloring where every color class induces an independent set or a clique

Subcoloring
: An improper vertex coloring where every color class induces a union of cliques

Weak coloring
: An improper vertex coloring where every non-isolated node has at least one neighbor with a different color

Sum-coloring
: The criterion of minimalization is the sum of colors

Coloring can also be considered for signed graphs and gain graphs.

See also

- Uniquely colorable graph
- Critical graph
- Graph homomorphism
- Mathematics of Sudoku

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External links

- *Graph Coloring Page* [48] by Joseph Culberson (graph coloring programs)
- [49] by Jim Andrews and Mike Fellows is a graph coloring puzzle
- GF-Graph Coloring Program [50]
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Multicategory

In mathematics (especially category theory), a **multicategory** is a generalization of the concept of category that allows morphisms of multiple arity. If morphisms in a category are viewed as analogous to functions, then morphisms in a multicategory are analogous to functions of several variables.

Definition

A multicategory consists of

- a collection (often a proper class) of *objects*;
- for every finite sequence (X_1, X_2, ..., X_n) of objects (for n := 0, 1, 2, ...) and object Y, a set of *morphisms* from X_1, X_2, ..., and X_n to Y; and
- for every object X, a special identity morphism (with n := 1) from X to X.

Additionally, there are composition operations: Given a sequence of sequences ($X_{1,1}$, $X_{1,2}$, ..., $X_{1,n1}$; $X_{2,1}$, $X_{2,2}$, ..., $X_{2,n2}$; ...; $X_{m,1}$, $X_{m,2}$, ..., $X_{m,n'm}$**) of objects, a sequence (**Y_1, Y_2, ..., Y_m**) of objects, and an object** Z**: if**

- f_1 is a morphism from $X_{1,1}$, $X_{1,2}$, ..., and $X_{1,n}$ to Y_1;
- f_2 is a morphism from $X_{2,1}$, $X_{2,2}$, ..., and $X_{2,n}$ to Y_2;
- ...;
- f_m is a morphism from $X_{m,1}$, $X_{m,2}$, ..., and $X_{m,n}$ to Y_m; and
- g is a morphism from Y_1, Y_2, ..., and Y_m to Z:

then there is a composite morphism $g(f_1, f_2, ..., f_m)$ from $X_{1,1}$, $X_{1,2}$, ..., $X_{1,n1}$, $X_{2,1}$, $X_{2,2}$, ..., $X_{2,n2}$, ..., $X_{m,1}$, $X_{m,2}$, ..., and $X_{m,n'm}$ **to** Z**. This must satisfy certain axioms:**

- If m is 1, Z is Y, and g is the identity morphism for Y, then $g(f)$ must equal f;
- if n_1 is 1, n_2 is 1, ..., n_m is 1, X_1 is Y_1, X_2 is Y_2, ..., X_m is Y_m, f_1 is the identity morphism for Y_1, f_2 is the identity morphism for Y_2, ..., and f_m is the identity morphism for Y_m, then $g(f_1, f_2, ..., f_m)$ must equal g; and
- an associativity condition (involving a further level of composition) that takes a long time to write down.

Examples

There is a multicategory whose objects are (small) sets, where a morphism from the sets X_1, X_2, ..., and X_n to the set Y is an n-ary function, that is a function from the Cartesian product $X_1 \times X_2 \times ... \times X_n$ to Y.

There is a multicategory whose objects are vector spaces (over the rational numbers, say), where a morphism from the vector spaces X_1, X_2, ..., and X_n to the vector space Y is a multilinear operator, that is a linear transformation from the tensor product $X_1 \otimes X_2 \otimes ... \otimes X_n$ to Y.

More generally, given any monoidal category **C**, there is a multicategory whose objects are objects of **C**, where a morphism from the **C**-objects X_1, X_2, ..., and X_n to the **C**-object Y is a **C**-morphism from the monoidal product of X_1, X_2, ..., and X_n to Y.

An operad is a multicategory with one unique object; except in degenerate cases, such a multicategory does not come from a monoidal category. (The term "operad" is often reserved for *symmetric* multicategories; terminology varies. [1])

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Enriched category

In category theory and its applications to mathematics, an **enriched category** is a category whose hom-sets are replaced by objects from some other category, in a well-behaved manner.

Definition

We define here what it means for **C** to be an **enriched category** over a monoidal category $(\mathbf{M}, \otimes, I)$.

The following structures are required:

- Let Ob(**C**) be a set (or proper class). An element of Ob(**C**) is called an *object* of **C**.
- For each pair (A,B) of objects of **C**, let Hom(A,B) be an object of **M**, called the *hom-object* of A and B.
- For each object A of **C**, let id_A be a morphism in **M** from I to Hom(A,A), called the *identity morphism* of A.
- For each triple (A,B,C) of objects of **C**, let

$$\circ : \mathrm{Hom}(B,C) \otimes \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,C)$$

be a morphism in **M** called the *composition* morphism of A, B, and C.

The following axioms are required:

- Associativity: Given objects A, B, C, and D of **C**, we can go from Hom(C,D) ⊗ Hom(B,C) ⊗ Hom(A,B) to Hom(A,D) in two ways, depending on which composition we do first. These must give the same result.

$$\begin{tikzcd}
(\mathrm{Hom}(C,D) \otimes \mathrm{Hom}(B,C)) \otimes \mathrm{Hom}(A,B) \arrow[r, "\alpha"] \arrow[d, "\circ\otimes 1"'] & \mathrm{Hom}(C,D) \otimes (\mathrm{Hom}(B,C) \otimes \mathrm{Hom}(A,B)) \arrow[r, "1\otimes\circ"] & \mathrm{Hom}(C,D) \otimes \mathrm{Hom}(A,C) \arrow[d, "\circ"] \\
\mathrm{Hom}(B,D) \otimes \mathrm{Hom}(A,B) \arrow[rr, "\circ"'] & & \mathrm{Hom}(A,D)
\end{tikzcd}$$

- Left identity: Given objects A and B of **C**, we can go from I ⊗ Hom(A,B) to just Hom(A,B) in two ways, either by using id_A on I and then using composition, or by simply using the fact that I is an identity for ⊗ in **M**. These must give the same result.
- Right identity: Given objects A and B of **C**, we can go from Hom(A,B) ⊗ I to just Hom(A,B) in two ways, either by using id_B on I and then using composition, or by simply using the fact that I is an identity for ⊗ in **M**. These must give the same result.

Given the above, **C** (consisting of all the structures listed above) is a category enriched over **M**.

Examples

The most straightforward example is to take **M** to be a category of sets, with the Cartesian product for the monoidal operation. Then **C** is nothing but an ordinary category. If **M** is the category of small sets, then **C** is a locally small category, because the hom-sets will all be small. Similarly, if **M** is the category of finite sets, then **C** is a locally finite category.

If **M** is the category **2** with Ob(**2**) = {0,1}, a single nonidentity morphism (from 0 to 1), and ordinary multiplication of numbers as the monoidal operation, then **C** can be interpreted as a preordered set. Specifically, $A \leq B$ iff Hom(A,B) = 1.

If **M** is a category of pointed sets with smash product for the monoidal operation, then **C** is a category with zero morphisms. Specifically, the zero morphism from A to B is the special point in the pointed set Hom(A,B).

If **M** is a category of abelian groups with tensor product as the monoidal operation, then **C** is a preadditive category.

Relationship with monoidal functors

If there is a monoidal functor from a monoidal category **M** to a monoidal category **N**, then any category enriched over **M** can be reinterpreted as a category enriched over **N**. Every monoidal category **M** has a monoidal functor **M**(I, –) to the category of sets, so any enriched category has an underlying ordinary category. In many examples (such as those above) this functor is faithful, so a category enriched over **M** can be described as an ordinary category with certain additional structure or properties.

Enriched functors

An **enriched functor** is the appropriate generalization of the notion of a functor to enriched categories. Enriched functors are then maps between enriched categories which respect the enriched structure.

If C and D are **M**-categories (that is, categories enriched over monoidal category **M**), an **M**-enriched functor T: $C \to D$ is a map which assigns to each object of C an object of D and for each pair of objects a and b in C provides a morphism in **M** T_{ab}: $C(a,b) \to D(T(a),T(b))$ between the hom-objects of C and D (which are objects in **M**), satisfying enriched versions of the axioms of a functor, viz preservation of identity and composition.

Because the hom-objects need not be sets in an enriched category, one cannot speak of a particular morphism. There is no longer any notion of an identity morphism, nor of a particular composition of two morphisms. Instead, morphisms from the unit to a hom-object should be thought of as selecting an identity and morphisms from the monoidal product should be thought of as composition. The usual functorial axioms are replaced with corresponding commutative diagrams involving these morphisms.

In detail, one has that the diagram

$$\begin{tikzcd} & I \arrow[dl, "\mathrm{id}_a"'] \arrow[dr, "\mathrm{id}_b"] & \\ C(a,a) \arrow[rr, "T_{aa}"] & & D(T(a),T(a)) \end{tikzcd}$$

commutes, which amounts to the equation

$$T_{aa} \circ \mathrm{id}_a = \mathrm{id}_{T(a)},$$

where I is the unit object of **M**. This is analogous to the rule $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ for ordinary functors. Additionally, one demands that the diagram

$$\begin{tikzcd}
C(b,c) \otimes C(a,b) \arrow[r, "\circ"] \arrow[d, "T_{bc} \otimes T_{ab}"'] & C(a,c) \arrow[d, "T_{ac}"] \\
D(T(b),T(c)) \otimes D(T(a),T(b)) \arrow[r, "\circ"'] & D(T(a),T(c))
\end{tikzcd}$$

commute, which is analogous to the rule $F(fg)=F(f)F(g)$ for ordinary functors.

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Metamathematics

Metatheory

A **metatheory** or **meta-theory** is a theory whose subject matter is some other theory. In other words it is a theory about a theory. Statements made in the metatheory about the theory are called metatheorems.

According to the systemic TOGA meta-theory [1] , a meta-theory may refer to the specific point of view on a theory and to its subjective meta-properties, but not to its application domain. In the above sense, a theory **T** of the domain **D** is a meta-theory if **D** is a theory or a set of theories. A general theory is not a meta-theory because its domain **D** are not theories.

The following is an example of a meta-theoretical statement:[2]

> Any physical theory is always provisional, in the sense that it is only a hypothesis; you can never prove it. No matter how many times the results of experiments agree with some theory, you can never be sure that the next time the result will not contradict the theory. On the other hand, you can disprove a theory by finding even a single observation that disagrees with the predictions of the theory.

Meta-theory belongs to the philosophical specialty of epistemology and metamathematics, as well as being an object of concern to the area in which the individual theory is conceived. An emerging domain of meta-theories is systemics.

Taxonomy

Examining groups of related theories, a first finding may be to identify classes of theories, thus specifying a taxonomy of theories. A proof engendered by a metatheory is called a *metatheorem*.

History

The concept burst upon the scene of twentieth-century philosophy as a result of the work of the German mathematician David Hilbert, who in 1905 published a proposal for proof of the consistency of mathematics, creating the field of metamathematics. His hopes for the success of this proof were dashed by the work of Kurt Gödel who in 1931 proved this to be unattainable by his incompleteness theorems. Nevertheless, his program of unsolved mathematical problems, out of which grew this metamathematical proposal, continued to influence the direction of mathematics for the rest of the twentieth century.

The study of metatheory became widespread during the rest of that century by its application in other fields, notably scientific linguistics and its concept of metalanguage.

See also

- meta-
- meta-knowledge
- Metalogic
- Metamathematics
- Philosophy of social science

External links

- Meta-theoretical Issues (2003), Lyle Flint [3]

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[2] Stephen Hawking in *A Brief History of Time*
[3] http://www.bsu.edu/classes/flint/comm360/metatheo.html

Metalogic

Metalogic is the study of the metatheory of logic. While *logic* is the study of the manner in which logical systems can be used to decide the correctness of arguments, metalogic studies the properties of the logical systems themselves.[1] According to Geoffrey Hunter, while logic concerns itself with the "truths of logic," metalogic concerns itself with the theory of "sentences used to express truths of logic."[2]

The basic objects of study in metalogic are formal languages, formal systems, and their interpretations. The study of interpretation of formal systems is the branch of mathematical logic known as model theory, while the study of deductive apparatus is the branch known as proof theory.

History

Metalogical questions have been asked since the time of Aristotle. However, it was only with the rise of formal languages in the late 19th and early 20th century that investigations into the foundations of logic began to flourish. In 1904, David Hilbert observed that in investigating the foundations of mathematics that logical notions are presupposed, and therefore a simultaneous account of metalogical and metamathematical principles was required. Today, metalogic and metamathematics are largely synonymous with each other, and both have been substantially subsumed by mathematical logic in academia.

Important distinctions in metalogic

Metalanguage-Object language

In metalogic, formal languages are sometimes called *object languages*. The language used to make statements about an object language is called a *metalanguage*. This distinction is a key difference between logic and metalogic. While logic deals with *proofs in a formal system*, expressed in some formal language, metalogic deals with *proofs about a formal system* which are expressed in a metalanguage about some object language.

Syntax-semantics

In metalogic, 'syntax' has to do with formal languages or formal systems without regard to any interpretation of them, whereas, 'semantics' has to do with interpretations of formal languages. The term 'syntactic' has a slightly wider scope than 'proof-theoretic', since it may be applied to properties of formal languages without any deductive systems, as well as to formal systems. 'Semantic' is synonymous with 'model-theoretic'.

Use-mention

In metalogic, the words 'use' and 'mention', in both their noun and verb forms, take on a technical sense in order to identify an important distinction.[2] The *use–mention distinction* (sometimes referred to as the *words-as-words distinction*) is the distinction between *using* a word (or phrase) and *mentioning* it. Usually it is indicated that an expression is being mentioned rather than used by enclosing it in quotation marks, printing it in italics, or setting the expression by itself on a line. The enclosing in quotes of an expression gives us the name of an expression, for example:

> 'Metalogic' is the name of this article.
>
> This article is about metalogic.

Type-token

The *type-token distinction* is a distinction in metalogic, that separates an abstract concept from the objects which are particular instances of the concept. For example, the particular bicycle in your garage is a token of the type of thing known as "The bicycle." Whereas, the bicycle in your garage is in a particular place at a particular time, that is not true of "the bicycle" as used in the sentence: "*The bicycle* has become more popular recently." This distinction is used to clarify the meaning of symbols of formal languages.

Overview

Formal language

A *formal language* is an organized set of symbols the essential feature of which is that it can be precisely defined in terms of just the shapes and locations of those symbols. Such a language can be defined, then, without any reference to any meanings of any of its expressions; it can exist before any interpretation is assigned to it -- that is, before it has any meaning. First order logic is expressed in some formal language. A formal grammar determines which symbols and sets of symbols are formulas in a formal language.

A formal language can be defined formally as a set A of strings (finite sequences) on a fixed alphabet α. Some authors, including Carnap, define the language as the ordered pair $<\alpha, A>$.[3] Carnap also requires that each element of α must occur in at least one string in A.

Formation rules

Formation rules (also called *formal grammar*) are a precise description of a the well-formed formulas of a formal language. It is synonymous with the set of strings over the alphabet of the formal language which constitute well formed formulas. However, it does not describe their semantics (i.e. what they mean).

Formal systems

A *formal system* (also called a *logical calculus*, or a *logical system*) consists of a formal language together with a deductive apparatus (also called a *deductive system*). The deductive apparatus may consist of a set of transformation rules (also called *inference rules*) or a set of axioms, or have both. A formal system is used to derive one expression from one or more other expressions.

A *formal system* can be formally defined as an ordered triple <α, $\mathcal{I}$, $\mathcal{D}$ d>, where $\mathcal{D}$ d is the relation of direct derivability. This relation is understood in a comprehensive sense such that the primitive sentences of the formal system are taken as directly derivable from the empty set of sentences. Direct derivability is a relation between a sentence and a finite, possibly empty set of sentences. Axioms are laid down in such a way that every first place member of $\mathcal{D}$ d is a member of $\mathcal{I}$ and every second place member is a finite subset of $\mathcal{I}$.

It is also possible to define a *formal system* using only the relation $\mathcal{D}$ d. In this way we can omit $\mathcal{I}$, and α in the definitions of *interpreted formal language*, and *interpreted formal system*. However, this method can be more difficult to understand and work with.[3]

Formal proofs

A *formal proof* is a sequences of well-formed formulas of a formal language, the last one of which is a theorem of a formal system. The theorem is a syntactic consequence of all the well formed formulae preceding it in the proof. For a well formed formula to qualify as part of a proof, it must be the result of applying a rule of the deductive apparatus of some formal system to the previous well formed formulae in the proof sequence.

Interpretations

An *interpretation* of a formal system is the assignment of meanings, to the symbols, and truth-values to the sentences of the formal system. The study of interpretations is called Formal semantics. *Giving an interpretation* is synonymous with *constructing a model.*

Results in metalogic

Results in metalogic consist of such things as formal proofs demonstrating the consistency, completeness, and decidability of particular formal systems.

Major results in metalogic include:

- Proof of the uncountability of the set of all subsets of the set of natural numbers (Cantor's theorem 1891)
- Löwenheim-Skolem theorem (Leopold Löwenheim 1915 and Thoralf Skolem 1919)
- Proof of the consistency of truth-functional propositional logic (Emil Post 1920)
- Proof of the semantic completeness of truth-functional propositional logic (Paul Bernays 1918)[4] ,(Emil Post 1920)[2]
- Proof of the syntactic completeness of truth-functional propositional logic (Emil Post 1920)[2]
- Proof of the decidability of truth-functional propositional logic (Emil Post 1920)[2]
- Proof of the consistency of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the semantic completeness of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the decidability of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the consistency of first order predicate logic (David Hilbert and Wilhelm Ackermann 1928)

- Proof of the semantic completeness of first order predicate logic (Gödel's completeness theorem 1930)
- Proof of the undecidability of first order predicate logic (Church's theorem 1936)
- Gödel's first incompleteness theorem 1931
- Gödel's second incompleteness theorem 1931

See also

- Metamathematics

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[4] Hao Wang, Reflections on Kurt Gödel

Metamathematics

Metamathematics is the study of mathematics itself using mathematical methods. This study produces metatheories, which are mathematical theories about other mathematical theories. Metamathematical metatheorems about mathematics itself were originally differentiated from ordinary mathematical theorems in the 19th century, to focus on what was then called the foundational crisis of mathematics. Richard's paradox (Richard 1905) concerning certain 'definitions' of real numbers in the English language is an example of the sort of contradictions which can easily occur if one fails to distinguish between mathematics and metamathematics.

The term "metamathematics" is sometimes used as a synonym for certain elementary parts of formal logic, including propositional logic and predicate logic.

History

Metamathematics was intimately connected to mathematical logic, so that the early histories of the two fields, during the late 19th and early 20th centuries, largely overlap. More recently, mathematical logic has often included the study of new pure mathematics, such as set theory, recursion theory and pure model theory, which is not directly related to metamathematics.

Serious metamathematical reflection began with the work of Gottlob Frege, especially his *Begriffsschrift*.

David Hilbert was the first to invoke the term "metamathematics" with regularity (see Hilbert's program). In his hands, it meant something akin to contemporary proof theory, in which finitary methods are used to study various axiomatized mathematical theorems.

Other prominent figures in the field include Bertrand Russell, Thoralf Skolem, Emil Post, Alonzo Church, Stephen Kleene, Willard Quine, Paul Benacerraf, Hilary Putnam, Gregory Chaitin, Alfred Tarski and Kurt Gödel. In particular, Gödel's proof that, given any finite number of axioms for Peano arithmetic, there will be true statements about that arithmetic that cannot be proved from those axioms, a result known as the incompleteness theorem, is arguably the greatest achievement of metamathematics and the philosophy of mathematics to date.

Milestones

- Principia Mathematica (Whitehead and Russell 1925)
- Gödel's completeness theorem, 1930
- Gödel's incompleteness theorem, 1931
- Tarski's definition of model-theoretic satisfaction, now called the T-schema
- The proof of the impossibility of the Entscheidungsproblem, obtained independently in 1936–1937 by Church and Turing.

See also

- Meta-
- Model theory
- Philosophy of mathematics
- Proof theory

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Alexander Grothendieck

Alexander Grothendieck	
 Alexander Grothendieck in Montreal, 1970	
Born	28 March 1928 Berlin, Germany
Residence	France
Fields	algebraic geometry, homological algebra, and functional analysis
Alma mater	University of Montpellier, University of Nancy
Doctoral advisor	Laurent Schwartz
Doctoral students	Pierre Deligne, Jean-Louis Verdier, Michel Raynaud
Notable awards	Fields Medal (1966), Crafoord Prize(1988, declined)

Alexander Grothendieck (born March 28, 1928 in Berlin, Germany) is one of the most influential mathematicians of the twentieth century, known principally for his revolutionary advances in algebraic geometry, but also for major contributions to algebraic topology, number theory, category theory, Galois theory, descent theory, commutative homological algebra and functional analysis. He was awarded the Fields Medal in 1966, and was co-awarded the Crafoord Prize with Pierre Deligne in 1988, but Grothendieck declined it.

He is noted for his mastery of abstract approaches to mathematics, and his perfectionism in matters of formulation and presentation. In particular, he demonstrated the ability to derive concrete results using only very general methods. Relatively little of his work after 1960 was published by the conventional route of the learned journal, circulating initially in duplicated volumes of seminar notes; his influence was to a considerable extent personal, on French mathematics and the Zariski school at Harvard University. He retired in 1988 and within a few years became reclusive.

Mathematical achievements

Grothendieck's early mathematical work was done in functional analysis between 1949 and 1953 working on (what became) his doctoral thesis in Nancy, supervised by Jean Dieudonné and Laurent Schwartz. His key contributions include topological tensor products of vector spaces, the theory of nuclear spaces and the application of L^p spaces in studying linear maps between topological vector spaces. In the space of a few years, he had turned himself into a leading authority on the theory of topological vector spaces — to the extent that Dieudonné compares his impact in this field to that of Banach.[1]

However, it is algebraic geometry and related fields where Grothendieck did his most important and influential work. From about 1955 he started to work on sheaf theory and homological algebra, rapidly producing the very influential "Tôhoku paper" (*Sur quelques points d'algèbre homologique*, published in 1957) where he introduced Abelian categories and applied it to show that sheaf cohomology can be defined as certain derived functors in this context.

Homological methods and sheaf theory had already been introduced in algebraic geometry by Jean-Pierre Serre and others, after sheaves had been defined by Jean Leray. Grothendieck took them to a higher level of abstraction and turned them into a key organising principle of his theory. He thereby changed the tools and the level of abstraction in algebraic geometry. He shifted attention from the study of individual varieties to the *relative point of view* (pairs of varieties related by a morphism), allowing a broad generalization of many classical theorems. The first major application was the relative version of Serre's theorem showing that the cohomology of a coherent sheaf on a complete variety is finite dimensional; Grothendieck's theorem shows that the higher direct images of coherent sheaves under a proper map are coherent; this reduces to Serre's theorem over a one-point space.

Next, in 1956, he applied the same thinking to the Riemann–Roch theorem, which had already recently been generalized to any dimension by Hirzebruch. The Grothendieck–Riemann–Roch theorem was announced by Grothendieck at the initial Mathematische Arbeitstagung in Bonn, in 1957. It appeared in print in a paper written by Armand Borel with Serre. This result was his first major achievement in algebraic geometry. He went on to plan and execute a major foundational programme for rebuilding the foundations of algebraic geometry; he exposed the main outlines of this programme in his talk at the 1958 International Congress of Mathematicians.

His foundational work on algebraic geometry is at a higher level of abstraction than all prior versions. He adapted the use of non-closed generic points, which led to the theory of schemes. He also pioneered the systematic use of nilpotents. As 'functions' these can take only the value 0, but they carry infinitesimal information, in purely algebraic settings. His *theory of schemes* has become established as the best universal foundation for this major field, because of its great expressive power as well as technical depth. In that setting one can use birational geometry, techniques from number theory, Galois theory and commutative algebra, and close analogues of the methods of algebraic topology, all in an integrated way.[2] [3] [4]

His influence spilled over into many other branches of mathematics, for example the contemporary theory of D-modules. (It also provoked adverse reactions, with many mathematicians seeking out more concrete areas and problems.)[5] [6]

EGA and SGA

The bulk of Grothendieck's published work is collected in the monumental, and yet incomplete, *Éléments de géométrie algébrique* (EGA) and *Séminaire de géométrie algébrique* (SGA). The collection *Fondements de la Géometrie Algébrique* (FGA), which gathers together talks given in the Séminaire Bourbaki, also contains important material.

Perhaps Grothendieck's deepest single accomplishment is the invention of the étale and l-adic cohomology theories, which explain an observation of André Weil's that there is a deep connection between the topological characteristics of a variety and its diophantine (number theoretic) properties. For example, the number of solutions of an equation over a finite field reflects the topological nature of its solutions over the complex numbers. Weil realized that to prove such a connection one needed a new cohomology theory, but neither he nor any other expert saw how to do this until such a theory was found by Grothendieck.

This program culminated in the proofs of the Weil conjectures, the last of which was settled by Grothendieck's student Pierre Deligne in the early 1970s after Grothendieck had largely withdrawn from mathematics.

Major mathematical topics (from *Récoltes et Semailles*)

He wrote a retrospective assessment of his mathematical work (see the external link *La Vision* below). As his main mathematical achievements ("maître-thèmes"), he chose this collection of 12 topics (his chronological order):

1. Topological tensor products and nuclear spaces
2. "Continuous" and "discrete" duality (derived categories and "six operations").
3. *Yoga* of the Grothendieck–Riemann–Roch theorem (K-theory, relation with intersection theory).
4. Schemes.
5. Topoi.
6. Étale cohomology including l-adic cohomology.
7. Motives and the motivic Galois group (and Grothendieck categories)
8. Crystals and crystalline cohomology, *yoga* of De Rham and Hodge coefficients.
9. Topological algebra, infinity-stacks, 'dérivateurs', cohomological formalism of toposes as an inspiration for a new homotopic algebra
10. Tame topology.
11. *Yoga* of anabelian geometry and Galois–Teichmüller theory.
12. Schematic point of view, or "arithmetics" for regular polyhedra and regular configurations of all sorts.

He wrote that the central theme of the topics above is that of topos theory, while the first and last were of the least importance to him.

Here the term *yoga* denotes a kind of "meta-theory" that can be used heuristically. The word *yoke*, meaning "linkage", is derived from the same Indo-European root.

Life

Family and early life

Alexander Grothendieck was born in Berlin to anarchist parents: a Russian father from an ultimately Hassidic family, Alexander "Sascha" Shapiro aka Tanaroff, and a mother from a German Protestant family, Johanna "Hanka" Grothendieck; both of his parents had broken away from their early backgrounds in their teens.[7] At the time of his birth Grothendieck's mother was married to Johannes Raddatz, a German journalist, and his birthname was initially recorded as *Alexander Raddatz*. The marriage was dissolved in 1929 and Shapiro/Tanaroff acknowledged his paternity, but never married Hanka Grothendieck.[7] Grothendieck lived with his parents until 1933 in Berlin. At the end of that year, Shapiro moved to Paris, and Hanka followed him the next year. They left Grothendieck in the care of Wilhelm Heydorn, a Lutheran Pastor and teacher[8] in Hamburg where he went to school. During this time, his parents fought in the Spanish Civil War.

During WWII

In 1939 Grothendieck came to France and lived in various camps for displaced persons with his mother, first at the Camp de Rieucros, and subsequently lived for the remainder of the war in the village of Le Chambon-sur-Lignon, where he, along with other Jewish children, was sheltered and hidden in local boarding-houses or pensions. His father was sent via Drancy to Auschwitz where he died in 1942. While Grothendieck lived in Chambon, he attended the Collège Cévenol (now known as the Le Collège-Lycée Cévenol International), a unique secondary school founded in 1938 by local Protestant pacifists and anti-war activists. Many of the other Jewish refugee children being hidden in Chambon attended Cévenol and it was at this school that Grothendieck apparently first became fascinated with mathematics.

Studies and contact with research mathematics

After the war, the young Grothendieck studied mathematics in France, initially at the University of Montpellier. He had decided to become a math teacher because he had been told that mathematical research had been completed early in the 20th century and there were no more open problems.[9] However, his talent was noticed, and he was encouraged to go to Paris in 1948.

Initially, Grothendieck attended Henri Cartan's Seminar at École Normale Supérieure, but lacking the necessary background to follow the high-powered seminar, he moved to the University of Nancy where he wrote his dissertation under Laurent Schwartz in functional analysis, from 1950 to 1953. At this time he was a leading expert in the theory of topological vector spaces. By 1957, he set this subject aside in order to work in algebraic geometry and homological algebra.

The IHÉS years

Installed at the Institut des Hautes Études Scientifiques (IHÉS), Grothendieck attracted attention by an intense and highly productive activity of seminars (*de facto* working groups drafting into foundational work some of the ablest French and other mathematicians of the younger generation). Grothendieck himself practically ceased publication of papers through the conventional, learned journal route. He was, however, able to play a dominant role in mathematics for around a decade, gathering a strong school.

During this time he had officially as students Michel Demazure (who worked on SGA3, on group schemes), Luc Illusie (cotangent complex), Michel Raynaud, Jean-Louis Verdier (cofounder of the derived category theory) and Pierre Deligne. Collaborators on the SGA projects also included Mike Artin (étale cohomology) and Nick Katz (monodromy theory and Lefschetz pencils). Jean Giraud worked out torsor theory extensions of non-abelian cohomology. Many others were involved.

The 'Golden Age'

Alexander Grothendieck's work during the `Golden Age' period at IHÉS established several unifying themes in algebraic geometry, number theory, topology, category theory and complex analysis. His first (pre-IHÉS) breakthrough in algebraic geometry was the Grothendieck–Hirzebruch–Riemann–Roch theorem, a far-reaching generalisation of the Hirzebruch–Riemann–Roch theorem proved algebraically; in this context he also introduced K-theory. Then, following the programme he outlined in his talk at the 1958 International Congress of Mathematicians, he introduced the theory of schemes, developing it in detail in his Éléments de géométrie algébrique (EGA) and providing the new more flexible and general foundations for algebraic geometry that has been adopted in the field since that time. He went on to introduce the étale cohomology theory of schemes, providing the key tools for proving the Weil conjectures, as well as crystalline cohomology and algebraic de Rham cohomology to complement it. Closely linked to these cohomology theories, he originated topos theory as a generalisation of topology (relevant also in mathematical logic, category theory, and also to computer software programming and institutional ontology classification and bioinformatics). He also provided an algebraic definition of fundamental groups of schemes and more generally the main structures of a categorical Galois theory. As a framework for his coherent duality theory he also introduced derived categories, which were further developed by Verdier.

The results of work on these and other topics were published in the EGA and in less polished form in the notes of the Séminaire de géométrie algébrique (SGA) that he directed at IHES.

Politics and retreat from scientific community

Grothendieck's political views were radical and pacifist. Thus he strongly opposed both United States aggression in Vietnam as well as Soviet military expansionism. He gave lectures on category theory in the forests surrounding Hanoi while the city was being bombed, to protest against the Vietnam War (*The Life and Work of Alexander Grothendieck*, American Math. Monthly, vol. 113, no. 9, footnote 6). He retired from scientific life around 1970, after having discovered the partly military funding of IHÉS (see pp. xii and xiii of SGA1, Springer Lecture Notes 224). He returned to academia a few years later as a professor at the University of Montpellier, where he stayed until his retirement in 1988. His criticisms of the scientific community, and especially of several mathematics circles, are also contained in a letter, written in 1988, in which he states the reasons for his refusal of the Crafoord Prize.[10] He declined the prize on ethical grounds in an open letter to the media.[11]

While the issue of military funding was perhaps the most obvious explanation for Grothendieck's departure from IHÉS, those who knew him say that the causes of the rupture ran deeper. Pierre Cartier, a *visiteur de longue durée* ("long-term guest") at the IHÉS, wrote a piece about Grothendieck for a special volume published on the occasion of the IHÉS's fortieth anniversary. The *Grothendieck Festschrift* was a three-volume collection of research papers to mark his sixtieth birthday (falling in 1988), and published in 1990.[12]

In it Cartier notes that, as the son of an antimilitary anarchist and one who grew up among the disenfranchised, Grothendieck always had a deep compassion for the poor and the downtrodden. As Cartier puts it, Grothendieck came to find Bures-sur-Yvette "*une cage dorée*" ("a golden cage"). While Grothendieck was at the IHÉS, opposition to the Vietnam War was heating up, and Cartier suggests that this also reinforced Grothendieck's distaste at having become a mandarin of the scientific world. In addition, after several years at the IHÉS Grothendieck seemed to cast about for new intellectual interests. By the late 1960s he had started to become interested in scientific areas outside of mathematics. David Ruelle, a physicist who joined the IHÉS faculty in 1964, said that Grothendieck came to talk to him a few times about physics. (In the 1970s Ruelle and the Dutch mathematician Floris Takens produced a new model for turbulence, and it was Ruelle who invented the concept of a strange attractor in a dynamical system.) Biology interested Grothendieck much more than physics, and he organized some seminars on biological topics.[13]

After leaving the IHÉS, Grothendieck tried but failed to get a position at the Collège de France.

He then went to Université de Montpellier, where he became increasingly estranged from the mathematical community. Around this time, he founded a group called Survivre, which was dedicated to antimilitary and ecological issues. His mathematical career, for the most part, ended when he left the IHÉS. In 1984 he wrote a proposal to get a position through the Centre National de la Recherche Scientifique. The proposal, entitled *Esquisse d'un Programme* ("Program Sketch") describes new ideas for studying the moduli space of complex curves. Although Grothendieck himself never published his work in this area, the proposal became the inspiration for work by other mathematicians and the source of the theory of dessin d'enfants. *Esquisse d’un Programme* was published in the two-volume proceedings Geometric Galois Actions (Cambridge University Press, 1997)."[14]

Manuscripts written in the 1980s

While not publishing mathematical research in conventional ways during the 1980s, he produced several influential manuscripts with limited distribution, with both mathematical and biographical content. During that period he also released his work on Bertini type theorems contained in EGA 5, published by the Grothendieck Circle [15] in 2004.

La Longue Marche à travers la théorie de Galois [*The Long March Through Galois Theory*] is an approximately 1600-page handwritten manuscript produced by Grothendieck during the years 1980–1981, containing many of the ideas leading to the *Esquisse d'un programme*[16] (see below, and also a more detailed entry), and in particular studying the Teichmüller theory. (For an English translation of the tables of contents of these manuscripts see the Wikipedia separate entry on the Esquisse d'un programme.)

In 1983 he wrote a huge extended manuscript (about 600 pages) entitled *Pursuing Stacks*, stimulated by correspondence with Ronald Brown, (see also R.Brown [17] and Tim Porter at University of Bangor in Wales), and starting with a letter addressed to Daniel Quillen. This letter and successive parts were distributed from Bangor (see External Links below): in an informal manner, as a kind of diary, Grothendieck explained and developed his ideas on the relationship between algebraic homotopy theory and algebraic geometry and prospects for a noncommutative theory of stacks. The manuscript, which is being edited for publication by G. Maltsiniotis, later led to another of his monumental works, *Les Dérivateurs*. Written in 1991, this latter opus of about 2000 pages further developed the homotopical ideas begun in *Pursuing Stacks*. Much of this work anticipated the subsequent development of the motivic homotopy theory of F. Morel and V. Voevodsky in the mid 1990s.

His *Esquisse d'un programme*[16] (1984) is a proposal for a position at the Centre National de la Recherche Scientifique, which he held from 1984 to his retirement in 1988. Ideas from it have proved influential, and have been developed by others, in particular dessins d'enfants and a new field emerging as anabelian geometry. In *La Clef des Songes* he explains how the reality of dreams convinced him of God's existence.

The 1000-page autobiographical manuscript *Récoltes et semailles* (1986) is now available on the internet in the French original, and an English translation is underway (these parts of Récoltes et semailles have already been translated into Russian] and published in Moscow[18]). Some parts of *Récoltes et semailles*[19] [20] and the whole *La Clef des Songes* [21] have been translated into Spanish.

Retirement into reclusion

In 1991, Grothendieck moved to an address he did not provide to his previous contacts in the mathematical community. He is now said to live in southern France or Andorra and to be reclusive.

See also

- Ax-Grothendieck theorem
- Birkhoff–Grothendieck theorem
- Esquisse d'un Programme
- Grothendieck category [22]
- Grothendieck's connectedness theorem
- Grothendieck connection
- Grothendieck construction
- Grothendieck's Galois theory
- Grothendieck group
- Grothendieck inequality or Grothendieck constant
- Grothendieck–Katz p-curvature conjecture
- Grothendieck's relative point of view
- Grothendieck–Riemann–Roch theorem
- Grothendieck's Séminaire de géométrie algébrique
- Grothendieck space
- Grothendieck spectral sequence
- Grothendieck topology
- Grothendieck universe
- Tarski–Grothendieck set theory
- IHES
- IHES at Forty by Allyn Jackson [23]

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External links

- O'Connor, John J.; Robertson, Edmund F., "Alexander Grothendieck" [34], *MacTutor History of Mathematics archive*, University of St Andrews.
- Alexander Grothendieck [35] at the Mathematics Genealogy Project
- Grothendieck Circle [36], collection of mathematical and biographical information, photos, links to his writings
- Institut des Hautes Études Scientifiques [37]
- The origins of `Pursuing Stacks' [38] This is an account of how `Pursuing Stacks' was written in response to a correspondence in English with Ronnie Brown and Tim Porter [39] at Bangor, which continued until 1991.
- Récoltes et Semailles [40] in French.
- Spanish translation [41] of "Récoltes et Semailles" et "Le Clef des Songes" and other Grothendieck's texts
- short bio [31] from Notices of the American Mathematical Society

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[7] Society for Industrial and Applied Mathematics (http://www.siam.org/news/news.php?id=1405)

[8] Allyn Jackson, The Life of Alexander Grothendieck, p. 1040 (http://www.ams.org/notices/200409/fea-grothendieck-part1.pdf)

[9] See Jackson (2004:1). The remark is from the beginning of Récoltes et Semailles (page P4, in the introductory section Prélude en quatre Mouvements)

[10] Crafoord Prize letter (http://web.archive.org/web/20060106062005/http://www.math.columbia.edu/~lipyan/CrafoordPrize.pdf)

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[12] The editors were Pierre Cartier, Luc Illusie, Nick Katz, Gérard Laumon, Yuri Manin, and Ken Ribet. A second edition has been printed (2007) by Birkhauser.

[13] The IHÉS at Forty (http://www.ams.org/notices/199903/ihes-changes.pdf) Allyn Jackson, March 1999, Noticed of the AMS pp. 329-337

[14] Google book link (http://books.google.co.uk/books?id=Hlf1j2XmXkcC&dq=Geometric+Galois+Actions&printsec=frontcover&source=bl&ots=q9iN4QMkYj&sig=fAsvVlPOhN9K9LfjGFJC5zm_IA4&hl=en&ei=KW-3ScSkHYHIMuDukOEK&sa=X&oi=book_result&resnum=1&ct=result)

[15] http://www.math.jussieu.fr/~leila/grothendieckcircle/index.php

[16] ESQUISSE D’UN PROGRAMME par Alexandre Grothendieck (http://matematicas.unex.es/~navarro/res/esquissefr.pdf)

[17] http://www.bangor.ac.uk/r.brown

[18] In Russian (http://www.mccme.ru/free-books/grothendieck/RS.html)

[19] COSECHAS Y SIEMBRAS: Reflexiones y testimonios sobre un pasado de matem´atico (http://matematicas.unex.es/~navarro/res/preludio.pdf) Preludio

[20] COSECHAS Y SIEMBRAS: Reflexiones y testimonios sobre un pasado de matem´atico (http://matematicas.unex.es/~navarro/res/carta.pdf) Carta

[21] La Clef des Songes (http://matematicas.unex.es/~navarro/res/clef1-6.pdf)

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[28] http://www.ams.org/notices/200410/fea-grothendieck-part2.pdf

[29] http://www.sciencenews.org/view/generic/id/31898/title/Sensitivity_to_the_harmony_of_things

[30] http://www.scharlau-online.de/ag_1.html

[31] http://www.ams.org/notices/200808/tx080800930p.pdf

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[36] http://www.grothendieckcircle.org/

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Grothendieck's Programme

Esquisse d'un Programme

"Esquisse d'un Programme" is a famous proposal for long-term mathematical research made by the German-born, French mathematician Alexander Grothendieck [1] . He pursued the sequence of logically linked ideas in his important project proposal from 1984 until 1988, but his proposed research continues to date to be of major interest in several branches of advanced mathematics. Grothendieck's vision provides inspiration today for several developments in mathematics such as the extension and generalization of Galois theory, which is currently being extended based on his original proposal.

Brief history

Submitted in 1984, the *Esquisse d'un Programme*[2] was a successful proposal submitted by Alexander Grothendieck for a position at the Centre National de la Recherche Scientifique, which Grothendieck held from 1984 till 1988.[3] This proposal was however not formally published until 1997, because the author "could not be found, much less his permission requested".[4] The dessins d'enfants, or "children's drawings" and "anabelian algebraic geometry" — non-Abelian algebraic topology and noncommutative geometry — that are contained in this long-term program, continue even today to inspire extensive mathematical studies.

Abstract of Grothendieck's programme

("*Sommaire*")

- 1. The Proposal and enterpise ("Envoi").
- 2. "Teichmüller's Lego-game and the Galois group of Q over Q" ("Un jeu de “Lego-Teichmüller” et le groupe de Galois de Q sur Q").
- 3. Number fields associated with "dessin d'enfants". ("Corps de nombres associés à un dessin d’enfant").
- 4. Regular polyhedra over finite fields ("Polyèdres réguliers sur les corps finis").
- 5. General topology or a 'Moderated Topology' ("Haro sur la topologie dite 'générale', et réflexions heuristiques vers une topologie dite 'modérée").
- 6. Differentiable theories and moderated theories ("Théories différentiables" (à la Nash) et “théories modérées").
- 7. Pursuing Stacks ("À la Poursuite des Champs"}[5] .
- 8. Two-dimensional geometry ("Digressions de géométrie bidimensionnelle"[6] .
- 9. Summary of proposed studies ("Bilan d’une activité enseignante").
- 10. Epilogue.
- Notes

Suggested further reading for the interested mathematical reader is provided in the *References* section.

"The Long March across the Theory of Galois"

“This manuscript, consisting of some nearly 800 hand-written double pages, dating from 1981, was left behind with Grothendieck's other unpublished manuscripts when he disappeared in 1991. Typed in Tex, it comes out to about 400 pages. It goes together with a further 1,000 pages or so of additional notes and sections which have not yet been read or typed. Many of the major themes were summarised in the 1983 manuscript Esquisse d'un Programme.”

The Table of Contents for this important work by Alexander Grothendieck was originally compiled in French by the author and is reproduced here after the English Translation of the major parts of the Long March.

Table of Contents for the Long March across Galois Theory

Extensions of Galois's theory for groups: Galois groupoids, categories and functors

Galois has developed a powerful, fundamental algebraic theory in mathematics that provides very efficient computations for certain algebraic problems by utilizing the algebraic concept of groups, which is now known as the theory of Galois groups; such computations were not possible before, and also in many cases are much more effective than the 'direct' calculations without using groups[7] . To begin with, Alexander Grothendieck stated in his proposal: *"Thus, the group of Galois is realized as the automorphism group of a concrete, pro-finite group which respects certain structures that are essential to this group."* This fundamental, Galois group theory in mathematics has been considerably expanded, at first to groupoids- as proposed in Alexander Grothendieck's *Esquisse d' un Programme* (*EdP*)- and now already partially carried out for groupoids; the latter are now further developed beyond groupoids to categories by several groups of mathematicians. Here, we shall focus only on the well-established and fully validated extensions of Galois' theory. Thus, EdP also proposed and anticipated, along previous Alexander Grothendieck's *IHÉS* seminars (SGA1 to SGA4) held in the 1960s, the development of even more powerful extensions of the original Galois's theory for groups by utilizing categories, functors and natural transformations, as well as further expansion of the manifold of ideas presented in Alexander Grothendieck's *Descent Theory*. The notion of motive has also been pursued actively [8] . This was developed into the motivic Galois group, Grothendieck topology and Grothendieck category [9] . Such developments were recently extended in algebraic topology *via* representable functors and the fundamental groupoid functor.

See also

- Alexander Grothendieck
- Grothendieck's Galois theory
- Grothendieck group
- Grothendieck category [10]
- Grothendieck inequality or Grothendieck constant
- Grothendieck–Katz p-curvature conjecture
- Grothendieck's relative point of view
- Grothendieck–Riemann–Roch theorem
- Grothendieck's Séminaire de géométrie algébrique
- Grothendieck space
- Grothendieck spectral sequence
- Grothendieck topology
- Grothendieck universe
- Tarski–Grothendieck set theory
- IHES
- IHES at Forty by Allyn Jackson [23]

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External links

- Fundamental Groupoid Functors (http://planetphysics.org/encyclopedia/QuantumFundamentalGroupoid3.html), Planet Physics.

Grothendieck's Galois theory

In mathematics, **Grothendieck's Galois theory** is a highly abstract approach to the Galois theory of fields, developed around 1960 to provide a way to study the fundamental group of algebraic topology in the setting of algebraic geometry. It provides, in the classical setting of field theory, an alternative perspective to that of Emil Artin based on linear algebra, which became standard from about the 1930s.

The approach of Alexander Grothendieck is concerned with the category-theoretic properties that characterise the categories of finite *G*-sets for a fixed profinite group *G*. For example, *G* might be the group denoted $\hat{\mathbb{Z}}$, which is the inverse limit of the cyclic additive groups **Z**/n**Z** — or equivalently the completion of the infinite cyclic group **Z** for the topology of subgroups of finite index. A finite *G*-set is then a finite set *X* on which *G* acts through a quotient finite cyclic group, so that it is specified by giving some permutation of *X*.

In the above example, a connection with classical Galois theory can be seen by regarding $\hat{\mathbb{Z}}$ as the profinite Galois group Gal(F/F) of the algebraic closure F of any finite field *F*, over *F*. That is, the automorphisms of F fixing *F* are described by the inverse limit, as we take larger and larger finite splitting fields over *F*. The connection with geometry can be seen when we look at covering spaces of the unit disk in the complex plane with the origin removed: the finite covering realised by the z^n map of the disk, thought of by means of a complex number variable *z*, corresponds to the subgroup *n*.**Z** of the fundamental group of the punctured disk.

The theory of Grothendieck, published in SGA1, shows how to reconstruct the category of *G*-sets from a *fibre functor* Φ, which in the geometric setting takes the fibre of a covering above a fixed base point (as a set). In fact there is an isomorphism proved of the type

$$G \cong \mathrm{Aut}(\Phi),$$

the latter being the group of automorphisms (self-natural equivalences) of Φ. An abstract classification of categories with a functor to the category of sets is given, by means of which one can recognise categories of *G*-sets for *G* profinite.

To see how this applies to the case of fields, one has to study the tensor product of fields. Later developments in topos theory make this all part of a theory of *atomic toposes*.

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- Borceux, F. and Janelidze, G., Cambridge University Press (2001). *Galois theories*, ISBN 0521803098 (This book introduces the reader to the Galois theory of Grothendieck, and some generalisations, leading to Galois groupoids.)

Notes on Grothendieck's Galois Theory http://arxiv.org/PS_cache/math/pdf/0009/0009145v1.pdf

Ronald Brown (mathematician)

Ronald Brown, MA, D.Phil Oxon, FIMA, Emeritus Professor (born January 4, 1935) is an English mathematician. He is best known for his many, substantial contributions to Higher Dimensional Algebra and non-Abelian Algebraic Topology [14][1] , involving groupoids, algebroids [2], category theory, categorical generalizations of Galois theory, and generalization of the van Kampen theorem to higher homotopy groupoids, [3] as well as for being one of the first openly gay mathematicians in modern academia. These include four fundamental books and textbooks: *Elements of Modern Topology*, *Topology: a geometric account of general topology, homotopy types, and the fundamental groupoid*[4] ,[5] , *Topology and Groupoids*, and *Nonabelian algebraic topology* [14] (in two volumes) that contain original and important results in algebraic topology that are hard to obtain from other sources[1] . His editorial contributions over many years have provided generous, expert help and international support to several generations of mathematicians in rapidly developing areas of higher dimensional algebra, non-Abelian algebraic topology, including Category Theory, non-Abelian and Abelian, Homology and Cohomology [6], and Higher Dimensional Homotopy [7] with applications. Brown's interest in the general topology of function spaces began in the early 1960s, when he introduced the notion of an *adequate and convenient category of topological spaces for homotopy theory*, thus stimulating a wide range of work on convenient categories. Moreover, the term 'Higher Dimensional Algebra' was introduced in a 1987 survey paper by Brown [8] , following from the earlier *higher dimensional group theory*[9] introduced in 1982; this area has been remarkably successful not only in applications in other areas of mathematics, but also in quantum physics and computer science. Such potential applications that were recently suggested are novel algebraic topology and category theory approaches to extended quantum symmetry through quantum groupoid representations [10] to locally-covariant quantum gravity [11] theories and symmetry breaking. Several of Dr. Brown's papers combine methods of double groupoids [3] with differential ideas on holonomy, leading to the development of higher order notions of 'flows', analogous to evolving systems in concurrency theory. He collaborated with Higgins since the 1970s, and also with several other coworkers afterwards, on crossed complexes and the related higher homotopy groupoids [3]. He then completed the studies on pure higher order category theory in a publication with F.A. Al-Agl and R. Steiner, on "Multiple categories: the equivalence between a globular and cubical approach" [12], published in *Advances in Mathematics*, **170** (2002) 71-118[13] .

His key scientific results in mathematics to date have included: homotopy double groupoids [3], double algebroids [14], cubical omega-groupoids with connections[15] , and last-but-not least, proofs of higher-homotopy generalized van Kampen theorems [16] in homotopy theory[17] .

Dr. Ronald Brown has 115 items listed on MathSciNet, has given numerous presentations at scientific meetings, and published over 30 articles and items on popularization and teaching of mathematics. Two books are now in print, and a third one is close to being completed with two coworkers. He published over 200 research papers and presentations at scientific meetings, including several monographs and four books.

Biography

Ronald Brown was born on January 4, 1935 in London, England. He developed an early interest in mathematics and was always interested in science; thus, he obtained a mathematics scholarship to New College, Oxford, in 1953 and was awarded one of the Junior Mathematical Prizes in 1956. He then studied algebraic topology at Oxford, supervised first by J.H.C. Whitehead, (died 1960), and then, when at Liverpool, he was supervised by M.G. Barratt. Brown's thesis was submitted in 1961, under the supervision of Professor M.G. Barratt, and was on the homotopy type of function spaces, and this led to a long term interest in the applications of what are now called monoidal closed categories. The particular interest in the general topology of function spaces led to the notion of a "category adequate and convenient for all purposes of topology", and in ref.[18] he suggested for this end the categories of Hausdorff k-spaces and continuous functions, or Hausdorff spaces and k-continuous functions, thus stimulating a wide range of work on convenient categories. In collaboration with Peter Booth in the 1970s he helped develop Booth's notion of fiber-wise mapping spaces, i.e. a function space in the category of topological spaces over a given space B,[19] . The writing of a textbook on basic general and algebraic topology from a geometric viewpoint [20] led to his development of a generalisation to the non-connected case of the van Kampen theorem for the fundamental group, and then the use of groupoids for an exposition of most of 1-dimensional homotopy theory he won number 1 math student in his 3rd grade class.

After two university teaching appointments at Liverpool and at Hull University, he settled in 1970 at Bangor University in Wales where he became an Emeritus Professor in 2001. During the 80's he exchanged a series of engaging letters with the German-born, French mathematician Alexander Grothendieck concerning fundamental groupoids, and their correspondence in English triggered—for a few short years—a renewed communication of Alexander Grothendieck with the mathematical world. Brown visited Université Louis Pasteur in Strasbourg as an Associate Visiting Professor during 1983 and 1984, and had fruitful excahnges with several other French mathematicians, as for example, on groupoids with Jean Pradines, a research associate of former Professor Charles Ehresmann, (one of the founding mathematicians of category theory--along with Alexander Grothendieck—in France).

This suggested in 1965 the possibility of the existence and use of "higher homotopy groupoids", finally realised in a sequence of 12 papers by R. Brown and P.J. Higgins from 1978 to 2003, for which a recent survey is presented in [21] , and in a different form by R. Brown and J.-L. Loday in two papers in 1987,[22]

The idea from 1965 that these generalisations to higher dimensions of the non-Abelian fundamental groupoid should be developed in the spirit of group theory led to the term "higher dimensional group theory" [23] in 1982 and then to "higher dimensional algebra" in 1987 in the survey paper [24] . The applications to higher homotopy van Kampen theorems, which are in the area of 'local-to-global theorems', lead to some specific non-Abelian calculations in homotopy theory, for example of integral homotopy types, unavailable by other means, and to an understanding of certain homotopical ideas. The use of cubical methods in this work has also had applications in the use of algebraic and topological methods in the theory of concurrency in computer science. The investigation of "higher order symmetry" has also had applications to homotopy theory.[25] He has also worked on topological and differential groupoids, particularly with students, and the notion of holonomy and monodromy, pursuing ideas of Charles Ehresmann and J. Pradines. Working with T. Porter and A. Bak, Dr. Brown has developed the work of A. Bak on "global actions" to the notion of groupoid atlas, a kind of "algebraic patching" concept, and this has found applications in multiagent systems. Dr. Brown also has several papers in the area of symbolic computation and mathematical rewriting.

A long term interest in the popularization of mathematics led to a number of articles in this area [26], and to a collaboration in presenting the work of the sculptor John Robinson [27] .

Presently, in retirement, Professor Ronald Brown actively pursues his research in the beautiful surroundings of the village of Deganwy on the Conwy Estuary.

University education

· In 1956 B.A. at Oxford University . In 1961 Ph.D. at Liverpool University · In 1962 D.Phil. at Oxford University

Academic positions

· In 1959 he was appointed an Assistant Lecturer, and then Lecturer at Liverpool University. · During 1964–70 he worked as a Senior Lecturer, and then Reader at Hull University. · From 1970 to 1999 he taught and carried out research as a full Professor of Pure Mathematics at the University of Wales, Bangor, UK. · During 1970–1993 he functioned as the Head of Pure Mathematics, and also of the School of Mathematics in several variants · In 1990 he was elected as Chairman of the University of Wales Validation Board for a four year term · During 1983–84 he visited as a `Professeur associé pour un mois', at the Université Louis Pasteur in Strasbourg. · From 1999 to 2001 he was appointed a Half-time Research Professorship, and in September 2001 he became Professor Emeritus of the University of Wales.

Between 1959 and 2001 he advised 23 successful Ph.D. students in Mathematics.

Leading assignments

· 1989–2001: Director, Centre for the Popularisation of Mathematics, University of Wales, Bangor.

· 1995–2000: Coordinator, 'INTAS Project on Algebraic K-theory, groups and categories', for Bangor, the University of Bielefeld, Georgian Mathematical Institute, State Universities of Moscow and of St. Petersburg, and the Steklov Institute, St. Petersburg.

· 2002–2004 Leverhulme Emeritus Research Fellowship for a project on "Crossed complexes and homotopy groupoids".

Editorships

· Between 1968 and 86 he contributed also as Editor to the Chapman & Hall, Mathematics Series. · During 1975–1994 he was on the Editorial Advisory Board of the London Mathematical Society. · In 1995 he became a Founding member on the Management Committee of the Editorial Board of several electronic journals: *Theory and Applications of Categories.* · 1996–2007 Editorial Board: *Applied Categorical Structures* (Kluwer). · Since 1999 he is a Founding member of the electronic journal: *Homology, Homotopy and Applications*. 2006 — *Journal of Homotopy and Related Structures.*

Honors and awards

- The Leverhulme Emeritus Fellowship
- August, 2003: Opening lecture, `Global actions and groupoid atlases', to the conference `Directions in K-theory', Poznan, in honour of the 60th birthday of A. Bak.
- 2000: Grant to produce a CD-ROM as part of an EC Project *, 'Raising Public Awareness of Mathematics in WMY2000'.*
- 2003-2005: EPSRC Grant: Higher Dimensional algebra and Differential Geometry (Visiting Fellowship for J.F. Glazebrook, Eastern Illinois University, USA).

Selected publications

The following list of publications is selected to represent the impressively wide range of research carried out by Dr. Ronald Brown. For example his 1964 paper on "The twisted Eilenberg-Zilber theorem" became influential because it contained the first version of what is now known as the Homological Perturbation Lemma; the resulting Homological Perturbation Theory has afterwards proved to be an important theoretical and computational tool in algebraic topology and in the computation of resolutions.

- R. Brown. [Books 1, 2 and 3] *Elements of Modern Topology*, McGraw Hill, Maidenhead, (1968); second edition: *Topology: a geometric account of general topology, homotopy types, and the fundamental groupoid*, Ellis Horwood, Chichester (1988) 460 pp. Third edition: *Topology and Groupoids*, Booksurge LLC, (2006) xxv+525p.]
- R. Brown (with P.J. HIGGINS, R.SIVERA). [Book 4] *Nonabelian algebraic topology,* 2007 (vol.1), and vol.2 in 2008 (*in preparation*).
- R. Brown. Function spaces and product topologies, *Quart. J. Math*. (**2**) 15 (1964), 238-250. [**2**]
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- R. Brown (with P.I. BOOTH), On the application of fibred mapping spaces to exponential laws for bundles, ex-spaces and other categories of maps., *Gen. Top. Appl*. **8** (1978) 165–179.
- R.Brown (with J. HUEBSCHMANN), *Identities among relations*, in *Low dimensional topology, London Math. Soc. Lecture Note Series*, **48** (ed. R. Brown and T.L. Thickstun, Cambridge University Press) (1982), pp. 153–202. **This paper on identities among relations has been useful to many as a basic source.
- R.Brown (with S.P. HUMPHRIES), *Orbits under symplectic transvections II: the case K = F2*, *Proc. London Math. Soc*. (3) 52 (1986) 532–556.
- R.Brown (with P.J. HIGGINS), Tensor products and homotopies for omega-groupoids and crossed complexes, *J. Pure Appl. Alg*. **47** (1987) 1-33.
- R.Brown (with J.-L. LODAY), Homotopical excision, and Hurewicz theorems, for n-cubes of spaces, *Proc. London Math. Soc*. (**3**) 54 (1987) 176–192.
- R. Brown. From groups to groupoids: a brief survey, *Bull. London Math. Soc*., **19** (1987) 113-134. **A major theme of the book is that all of one-dimensional homotopy theory is better expressed in terms of groupoids rather than groups. This raised the question of applications of groupoids in higher homotopy theory, and so to a long march to higher order Van Kampen Theorems, which give new higher dimensional, non-Abelian, local-to-global methods, with relations to homology and K-theory.
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- R. Brown (with N.D. GILBERT)., Algebraic models of 3-types and automorphism structures for crossed modules, *Proc. London Math. Soc*. (3) **59** (1989) 51–73.
- R. Brown (with A. RAZAK SALLEH)., Free crossed resolutions of groups and presentations of modules of identities among relations, LMS J. Comp. and Math. *2 (1999) 28–61. Interest in algorithmic procedures and specific computations was shown in [107] and [124]. Such computations also occur in [51], which introduced a non-Abelian tensor product of groups which act on each other, and for which the bibliography now extends to over 100 papers.*
- R. Brown (with A. HEYWORTH)., Using rewriting systems to compute left Kan extensions and induced actions of categories, *J. Symbolic Computation* **29** (2000) 5–31.
- R. Brown (with I. IÇEN), Locally Lie subgroupoids and their Lie holonomy and monodromy groupoids, *Topology and its Applications*. **115** (2001) 125–138.

- R. Brown (with M. GOLASINSKI, T.PORTER and A.P.TONKS)., On function spaces of equivariant maps and the equivariant homotopy theory of crossed complexes II: the general topological group case., *K-Theory* **23** (2001) 129–155.
- R. Brown (with A. AL-AGL and R. STEINER)., Multiple categories: the equivalence between a globular and cubical approach, *Advances in Mathematics*, **170** (2002) 71–118.
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External links

- Ronald Brown's Home Page [36]
- Full list of Professor Ronald Brown's publications [37]
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